INFINITE GAMES WITH PERFECT INFORMATION

ZHANG WENZHANG

NATIONAL UNIVERSITY OF SINGAPORE

2006

INFINITE GAMES WITH PERFECT INFORMATION

ZHANG WENZHANG

(B.A. Beijing University)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ECONOMICS

NATIONAL UNIVERSITY OF SINGAPORE

2006

Acknowledgements

I would like to thank my supervisor, Professor Xiaolin Xing, for his many suggestions and constant support during this research.

The committee members, Dr. Younghwan In and Dr. Yohanes Eko Riyanto, have provided very helpful comments and suggestions.

The *University Research Scholarship*, which was awarded to me for the period 2004–2006, and the *President's Graduate Fellowship*, which was awarded to me for the period 2005–2006, were crucial to the completion of this thesis.

Singapore — Zhang Wenzhang

25 July, 2006

Contents

List of Figures

Summary

This thesis studies infinite games with perfect information, i.e., dynamic games in which players move sequentially.

Chapter 1 introduces the subject.

In Chapter 2 we consider the class of two-player perfect information games with characteristic payoff functions. We define a new solution concept for these games. The approach is axiomatic. We prove the determinacy for a whole class of perfect information games.

In Chapter 3 we introduce infinite perfect-information games with an infinite number of players. We also define a corresponding notion of determinacy for these games. An application to a simple overlapping generation model predicts monetary equilibrium as the only outcome of the economy.

In Chapter 4 we consider an effective version of determinacy for infinite games with perfect information by combining determinacy with the notion of computability by a Turing machine. We give a characterization of the determinacy for the class of closed games.

Chapter 1

Introduction

This thesis studies infinite games with perfect information, i.e., dynamic games in which players move sequentially. This class of games have been widely used in modeling economic activities. For example, Rubinstein (1982) uses an infinite two-person game with perfect information to settle the indeterminacy of bilateral bargaining over the gains from trade, Shaked and Sutton (1984) use infinite games to study involuntary unemployment and strike activity.

Subgame perfect Nash equilibrium is the most widely used solution concept in dynamic games. A fundamental difficulty in applying this concept is that it predicts a large number of equilibria in many games. For example it is known that there is a three-person bargaining game in which any outcome can be supported as subgame perfect Nash equilibrium.

The thesis focuses on refining subgame perfect Nash equilibrium for infinite games with perfect information.

1.1 The literature

The theory of infinite games has been approached from two quite different perspectives: the economic and the mathematical perspectives.

Mathematicians studied infinite games much earlier than economists. Gale and Stewart (1953) is the first systematic study of win-lose games. A two-player game is called a win-lose game if the payoffs of the players always sum up to 1. That is, in any play of the game, one and only one player wins. A win-lose game is called determined if one of the players has a winning strategy. Gale and Stewart (1953) proves the fundamental result that all games with closed or open payoff sets are determined. They also ask whether all games with Borel payoff sets are determined. After many years of studies (Wolfe (1955), Davis (1963)), this was finally confirmed by Martin (1975). The study of determinacy of win-lose games is also found to be closely related to the foundation of mathematics (see, e.g., Kanamori (2000)).

The studies of infinite games in economics starts with Rubinstein's seminal contribution to the bargaining problem. The settlement of various gains from trade is a fundamental problem in economics. Rubinstein (1982) approaches this problem through a bilateral bargaining process.

The story is told in the form of the division of a unite pie. Two players, 1 and 2, are bargaining over the partition of a pie. They take turns making proposals as to how it should be divided. In the first period player 1 proposes a partition, player 2 can either accept or reject this proposal; if he accepts then the game ends, otherwise they move to next period in which player 2 in turn proposes a partition to which player 1 replies; and so on.

In order for the players to have incentive to reach an agreement, certain assumptions on time preferences has to be imposed. In particular, if an agreement is never reached, each player gets nothing. Thus the bargaining process becomes an infinite game.

Rubinstein employs the notion of subgame perfect Nash equilibrium, which works both for finite and for infinite games (Selten (1965, 1975)), to study this game. The striking result is that under reasonable assumptions on time preferences the subgame perfect Nash equilibrium of this game is unique.

Attempts to generalize this result to the n-person case have been less successful. Shaked shows (reported in Sutton (1986) and Osborne and Rubinstein (1990)) that in a three-person bargaining game in which the players are sufficiently patient, any partition of the pie can be supported as a subgame perfect Nash equilibrium. But since then the use of infinite games in economics has been popular.

The mathematical studies have focused on zero-sum games and economists are interested only in non-zero-sum games. The studies in the economic side are mainly in the form of applications and examples. As far as we know, there is no systematic study of non-zero-sum games focusing on the infinite case.

Our study encompasses both perspectives. It is intended to be general and it covers the non-zero-sum games. We borrow techniques and terminologies from the mathematical literature and study games of interests both to economists and to mathematicians.

Detailed comparisons with related papers will be given in specific chapters. The rest of this chapter summaries the main results of each following chapter.

1.2 Structure of the Thesis

The main body of this thesis consists of three chapters: chapter 2, chapter 3 and chapter 4.

Chapter 2 and 3 study Non-zero-sum games. Chapter 2 deals with two player games with characteristic payoff functions. In chapter 3, the games under consideration are quite general: we allow arbitrary number, including infinitely many, of players and arbitrary payoff functions. The main purpose is to define a notion of determinacy for non-zero-sum games as a refinement of subgame perfect Nash equilibrium.

Chapter 4 studies two-person zero-sum games. The main purpose is to define an effective version of determinacy.

Chapter 4 can be read independently. It would be better to read chapter 2 before proceeding to chapter 3 although they are, strictly speaking, independent.

1.3 Summary of Chapter 2

Chapter 2 We consider the class of two-player perfect information games with characteristic payoff functions. Motivated by several simple examples in which subgame perfect Nash equilibrium fails to single out a unique equilibrium, we propose three behavioral axioms. Building on these axioms we develop a new solution concept termed determinacy. Intuitively, a game is determined if it can be solved by repeatedly applying these behavioral axioms. Moreover, determinacy turns out to be a unique refinement of subgame perfect Nash equilibria in the sense that the outcome of a determined game is always unique and can be supported by a subgame perfect Nash equilibrium. Closed games are games such that the underlying sets for the characteristic payoff functions are closed. We show that all closed games are determined. That is,determinacy solves a whole class of games.

1.4 Summary of Chapter 3

Chapter 3 introduces infinite perfect-information games with an infinite number of players. Many of the dynamic models in economics involve an infinite number of individuals. And interactions among the individuals play a crucial role there. Therefore in macroeconomics, quite often the model is infinite horizontal and infinite generations of individuals are involved. Moreover, the decisions of current generations have an impact on that of the future generations. Ideally we should formulate these situations using pure game-theoretic frameworks and apply the game-theoretic solution concepts to derive the relevant economic outcomes. But neither the framework of games with an infinite number of players is founded nor are appropriate solution concepts available.

In this paper, we develop the notion of a perfect information game with an infinite number of players. We also define a solution concept for this, namely the notions of determinacy, value and rational strategies. All these are natural extensions of the theory developed for infinite perfect-information games with a finite number of players.

We also present an application of this theory to a variant of Samuelson's Overlapping Generation Model. In contrast with the original derivation, this theory predicts the monetary equilibrium as the unique equilibrium.

1.5 Summary of Chapter 4

Chapter 4 defines an effective version of determinacy for infinite games with perfect information and characterizes the determinacy of closed games.

Turing machine has been one of the main tools in modeling bounded rationality in game theory. Intuitively, a Turing machine is a computer program that can be implemented by an ideal computer with no restriction on time and space. The main idea is that in many contexts, the players are not capable of playing strategies of arbitrary complexities. A natural idea of a strategy being simple is a strategy that is implementable by a Turing machine. Hence viewed from bounded rationality perspective, the class of Turing machine implementable strategies is a natural restriction on the strategies that can be used in a game.

Besides being a useful tool in modeling bounded rationality, there are practical uses of considering Turing machine implementable strategies. In many practical situations the agents playing the game, like computers, machines, robotics, are basically controlled by computer programs. So Turing machine implementable strategies are the right class of strategies to consider in these situations.

This chapter introduces an effective version of determinacy for infinite games with perfect information using the notion of computability (by Turing machines). We also give a characterization of effective determinacy for closed games.

Chapter 2

Games with Perfect Information

2.1 Introduction

Games with perfect information are dynamic games in which players move sequentially, i.e., no simultaneous moves. This chapter concerns the class of two-player perfect information games with characteristic payoff functions. That is, a player's payoff function can be represented by the characteristic function of a set: at the end of the game, if the sequence of choices lies in this set then the player's payoff is 1; otherwise his payoff is 0. In the former case we say that this player wins and he loses in the latter case. So at the end of a play a player either wins or loses; and it is possible that the players both win or both lose.

There are already several solution concepts applicable to this class of games in the literature. For the case of finite games, one can apply the backward induction algorithm (Zermelo (1913), Kuhn (1953)). In general, one can apply subgame perfect Nash equilibrium (Selten 1965, 1975), which has been the most widely used solution concept for more general games of perfect information. By insisting that in equilibrium the strategies players used should depend only on payoff relevant histories, one has the notion of Markov subgame perfect Nash equilibrium (see Maskin and Tirole (2001)).

Practice of these solution concepts have shown thatthough they are useful in many applications, there are still many other situations in which they are problematic. The most serious one is that they usually predict a large set of equilibria. This chapter constructs a different solution concept for the class of two player PI-games with characteristic payoff functions. It has the desired property of always predicting a unique payoff vector.

Our approach is axiomatic. We start with simple examples where it is clear enough what the solutions should be. Moreover, other solution concepts, such as backward induction or subgame perfect Nash equilibrium, fail to single out these intuitive solutions. Motivated by these examples, we propose axioms requiring the players to follow the intuitive solutions in these games. We then organize these axioms through iterations to define a new solution concept called determinacy.

2.1.1 An outline

We now describe the approach in more detail. We start with the axioms.

Our first axiom refines the usual backward induction. The usual backward induction solves (i.e., gives a unique prediction) only for the games in "general position". That is, the payoffs to each player at different leaves of the game tree are different. For other games backward induction may fail. We want to refine this backward induction algorithm so that it solves those games not in general position as well.

For an illustration, consider the following two-period game.

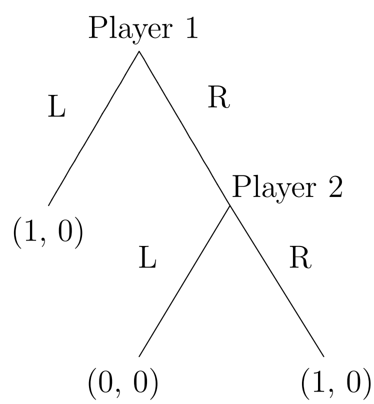

FIGURE 1

Since player 2's payoff of choosing L or R in the second period ties (both are 0), he is indifferent between which one to choose and both are possible. So according to the usual backward induction in the first period player 1 can choose either L or R since it is possible that choosing R could also lead to a payoff same as that of choosing L.

Thus the usual backward induction fails to give an instruction to player 1 as which one of the two choices to make in the first period. But it is easy to argue on an intuitive ground that L is a better choice. Suppose R is chosen by player 1, then in the second period player 2 may choose L so that player 1 loses the game since player 2 is indifferent between choosing L and R. So player 1 actually cannot guarantee a win in the subgame after R is played. But if, instead of playing R in the first period, he plays L then he is sure to win the game. So playing L gives him a much more secured winning situation than that of choosing R.

In order to rule out such irrational behaviors as player 1 choosing R in this game, we propose a refined backward induction as an axiom which, when applied to this game, requires the player 1 to choose L in the first period.

The following example motivates our second axiom. Two players, 1 and 2, alternate saying "stop" or "continue", starting with player 1. If either player says "stop", the game ends immediately and both get nothing; otherwise the play continues forever and each player receives one dollar.

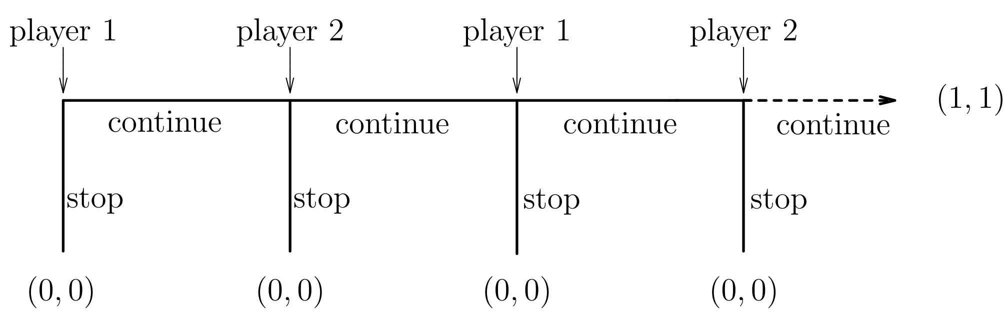

FIGURE 2

Although any path can be supported by a subgame perfect Nash equilibrium, the only reasonable one seems to be the path that players always say "continue". As long as there are chances for a player to win (to get the payoff of 1), why should he choose to lose for sure by saying "stop"? Moreover, it is common interest of both players to always say "continue" since that will give both the payoff of 1. So they can win the game by simply not "giving up".

In order to rule out the irrational moves such as saying "stop" in this game, we propose a cooperation axiom that requires the players always saying "continue", if it is applied to this game.

Our third axiom is a very natural, and even compelling, one. A strategy for a player is a winning strategy if following it he always wins the game regardless of how his opponent plays. The last axiom says that if a player has a winning strategy the he should follow it.

Building on these three axioms we develop the solution concept.

Say that a game is trivial if the players' payoff functions are constant, taking the value of either 0 or 1. That is, a player in a trivial game either always wins or always loses. A trivial game can be regarded as determined since no matter how the game is played the outcome is fixed and known.

Given an arbitrary game G, the axioms may not be directly applicable to it. But we can apply the axioms to its subgames. Say we apply an axiom to a subgame of G starting at a position p. Denote this subgame by G_p. Then by the axiom we know what the outcome of this subgame G_p should be. Say, the axiom predicts that it is a game both players lose. We then modify the game G by replacing the payoff of the subgame G_p by the constant functions taking value 0, i.e., the payoff functions of a trivial game in which both players lose. That is, we replace this subgame by a trivial game whose payoffs are that suggested by a behavioral axiom. So the payoff of G is changed since the payoff of the subgame G_p is changed. Denote this new game by G_1.

Since G and G_1 differ only in the subgame starting at the position p and the axiom suggests that both subgames have the same outcome. So from the pointview of the outcomes of games, G and G_1 are equivalent. But G_1 is simpler since, instead of having an arbitrary subgame, it has a trivial subgame. Intuitively, the reduction $G \rightarrow G_1$ partially solves G by solving a subgame of G.

We then iterate the reduction $G \rightarrow G_1$ by applying the axioms to G_1 or a subgame of G_1 to get a new game G_2. We can continue in this manner to obtain a chain $\langle G, G_1, G_2, \cdots, G_m \rangle$ of games. If at certain stage we reach a game G_n which is already trivial we then stop. Since the outcome of each G_k is same as that of G_{k+1} by the axioms, by an induction argument we know that the outcome of G is (the outcome of) this trivial game. In this way we then know what the outcome of G should be. We say such a game G is determined.

Summing up, a game G is determined if there exists a reduction chain $\langle G, G_1, G_2, \cdots, G_n \rangle$ with G_n trivial and each G_{k+1} is obtained from G_k by applying a behavior axiom to it. Intuitively, a game is determined if it can be "solved" by repeatedly solving many of its subgames, or, by collecting many partial solutions.

One of our main results is an existence type theorem. Say that a game is a closed game if the underlying sets for the characteristic payoff functions are closed sets. The theorem says that all closed games are determined. That is, determinacy as defined above solves the class of closed games.

2.1.2 The related literature

One way to put the present work in perspective is to recall the mathematical literature on the determinacy of win-lose games. For simplicity, call the class of games considered in this chapter non-zero-sum games. A game is called a win-lose game if the payoffs of the players always sum up to 1. That is, at any play of the game, one and only one player wins.

Gale and Stewart (1953) is the first systematic study of win-lose games. A win-lose game is called determined if one of the players has a winning strategy. Gale and Stewart (1953) proves the fundamental result that all games with closed or open payoff sets are determined. They also ask whether all games with Borel payoff sets are determined. After many years of studies (Wolfe (1955), Davis (1963)), this was finally confirmed by Martin (1975). The study of determinacy of win-lose games is also found to be closely related to the foundation of mathematics (see, e.g., Kanamori (2000)).

Our definition of determinacy extends the definition of determinacy in win-lose games to general non-zero-sum games. And the main theorem in this chapter extends that of Gale and Stewart (1953).

A second way to put our work in perspective is to recall the literature on equilibrium refinements. The aim of the equilibrium refinement program is to impose further behavioral criteria to reduce the set of Nash equilibrium and/or subgame perfect Nash equilibrium. Our work can be considered as a realization of this program in the case of two-player perfect-information games with characteristic payoff functions.

The basic axioms we propose refine subgame perfect Nash equilibrium in some very special games. Our first axiom refines backward induction, an important ingredient of subgame perfect Nash equilibrium. Our second axiom refines subgame perfect Nash equilibrium directly in a cooperative situation. Repeated applications of these axioms can be viewed as an iterated refining process. So determinacy is essentially an iteration of refinements. More formally, we show that the outcome of a determined game can be supported by a subgame perfect Nash equilibrium. Since the outcome of a determined game is unique, determinacy can be viewed as a unique refinement of subgame perfect Nash equilibria.

Detailed comparisons of determinacy with other solution concepts for games with perfect information will be given in Section 2.5.

The rest of this chapter is organized as follows. We give the formal definition of determinacy and main results in Section 2, postponing the axioms to Section 3. As an illustration, we prove the determinacy of all finite games of perfect information in Section 4. We then turn to a discussion of determinacy and the related literature in Section 5. All of the technical proofs are in Section 6.

2.2 Definitions and Main Result

We record here the notation and convention that will be used throughout the chapter.

Notation 2.2.1. 1. $\omega = \{0, 1, 2, \cdots\}$ denotes the set of natural numbers.

2. Y denotes an arbitrary set with at least two elements. E.g., $Y = \{0, 1\}$,

$Y = \{\mathrm{L}, \mathrm{R}\}$, $Y = \{\text{Continue}, \text{Stop}\}$, $Y = \omega$.

3. Y^ω denotes the set of all infinite sequences with elements from Y. Elements of Y^ω are denoted by f, g. E.g., $Y = \{\text{Continue}, \text{Stop}\}$, $f = (\text{Continue}, \text{Continue}, \text{Continue}, \cdots)$.

Definition 2.2.2. A game G is a pair $\langle A_1, A_2\rangle$, where $A_1, A_2 \subseteq Y^\omega$ are payoff sets of player 1 and 2 respectively.

The game $G = \langle A_1, A_2\rangle$ is interpreted as follows. Two players, player 1 and 2, alternate choosing elements from Y starting with player 1. Say player 1 first chooses y_0, then player 2, observing this, chooses y_1, player 1, after seeing y_1, makes a second choice y_2, etc. The game continues in this way so that an infinite sequence $(y_0, y_1, \cdots)$ is specified. Player i, $i = 1, 2$, wins just in case $(y_0, y_1, \cdots) \in A_i$.

Convention 2.2.3. If a statement or definition is for each of the two players, we shall only state it for player 1.

Definition 2.2.4. A game $G = \langle A_1, A_2\rangle$ is called a trivial game if each A_i is either Y^ω or $\emptyset$. If $G = \langle A_1, A_2\rangle$ is a trivial game, we say that player i wins G if A_i is Y^ω, otherwise he loses G.

So there are four trivial games: $\langle\emptyset, \emptyset\rangle$, $\langle\emptyset, Y^\omega\rangle$, $\langle Y^\omega, \emptyset\rangle$, and $\langle Y^\omega, Y^\omega\rangle$.

Definition 2.2.5. A reduction chain is a sequence $\langle G_0, G_1, \cdots, G_n\rangle$ of games such that each G_{k+1} is obtained from G_k by an application of the behavioral axioms to be defined in section 2.3.

Definition 2.2.6. A game $G = \langle A_1, A_2\rangle$ is called determined if there exists a reduction chain $\langle G_0, G_1, \cdots, G_n\rangle$ such that $G_0 = G$ and G_n is trivial.

Remark 2.2.7. For a technical reason Definition 2.2.6 is incomplete. But it suffices for the purpose of exposition of the main ideas. The technically complete version, which allows the chain to go infinitely long, is Definition 2.6.2 and 2.6.3. They will not be needed until Section 2.6.

The following theorem says that there is no ambiguity if there are more than one reductions. That is to say, if a game is determined then the outcome predicted by determinacy is unique.

Theorem 2.2.8. *Let $\langle G_0, G_1, \cdots, G_n\rangle$ and $\langle H_0, H_1, \cdots, H_m\rangle$ be two reduction chains for a game G such that $G_0 = H_0 = G$ and G_n, H_m are trivial. Then $G_n = H_m$.*

Remark 2.2.9. Another way of viewing this result is that it shows the consistency of the axiom system, in the sense that it never leads to contradicting conclusions.

Definition 2.2.10. Let $G = \langle A_1, A_2\rangle$ be a determined game with a reduction chain $\langle G_0, G_1, \cdots, G_n\rangle$. We say that player i wins the game G if he wins G_n, otherwise he lose G.

Let Y^ω be given the natural product topology with Y discrete, i.e., a basic open neighborhood is of the form

$$N_{(y_0, y_1, \cdots, y_m)} = \{(z_0, z_1, \cdots) \in Y^\omega \mid z_0 = y_0, \cdots, z_m = y_m\},$$

for each $(y_0, y_1, \cdots, y_m)$, where each y_k, $0 \leq k \leq m$, is an element of Y. A set $A \subset Y^\omega$ is an open set if it is a union of some basic open neighborhoods. $A \subset Y^\omega$ is said to be closed if its complement, $Y^\omega \setminus A$, is open.

Definition 2.2.11. A game $G = \langle A_1, A_2 \rangle$ is called a closed game if both A_1 and A_2 are closed sets.

The following is an existence-type theorem for determinacy.

Theorem 2.2.12. *All closed games are determined.*

Remark 2.2.13. Another way of viewing this result is that it says that the axiom system is complete, at least, for the class of closed games.

In Theorem 2.6.6 below, we shall show that the axioms are independent in the sense that if anyone of them is dropped, there will be a game that the remaining incomplete system is unable to solve.

Thus, we show that the axioms to be described in the following are consistent, complete and independent.

2.3 The Axioms

An axiom will have the following general format:

Axiom 2.3.1. A game $G = \langle A_1, A_2 \rangle$ satisfying certain conditions can be reduced to a game $G^* = \langle A_1^*, A_2^* \rangle$.

G^* will be G with the payoff of a subgame replaced by that of a trivial game. The axiom works as follows. As illustrated in the introduction, there is a behavioral assumption instructing how the players should play in this subgame. If we require the players to follow this assumption then we know what the outcome of this subgame is. The outcome will be one of the four trivial games. Then G^* is obtained from G by replacing this subgame by that trivial game suggested by the behavioral assumption. So although each axiom is a statement about reduction of games, there is actually an underlying behavioral assumption indicating what the rational players should do in a subgame.

Intuitively, the reduction $G \to G^*$ partially solves the game G by solving a subgame of G. Technically, G^* is topologically simpler and closer to a trivial game.

We need some following general definitions before proceeding.

Notation 2.3.2. 1. $Y^{<\omega}$ denotes the set of all finite sequences with elements from Y.

2. An element of $Y^{<\omega}$ will be denoted by p. E.g., $Y = \{0, 1\}$, $p = (1, 1, 1, 0, 0, 1)$; $Y = \{\text{Continue}, \text{Stop}\}$, $p = (\text{Continue}, \text{Continue}, \text{Stop})$.

3. The empty sequence, denoted by $\emptyset$, is in $Y^{<\omega}$. A sequence of length 1, $p = (y_0)$, will be identified with the element y_0.

4. Let $p = (y_0, y_1, \cdots, y_k) \in Y^{<\omega}$ and $y \in Y$, $p^\frown y$ denotes $(y_0, y_1, \cdots, y_k, y)$.

Definition 2.3.3. Let $p = (y_0, y_1, \cdots, y_n) \in Y^{<\omega}$. We say that p is a position for player 1 to move if n is an odd number or p is the empty sequence.

Definition 2.3.4. Let $G = \langle A_1, A_2 \rangle$ be a game and let $p = (y_0, y_1, \cdots, y_n) \in Y^{<\omega}$. The subgame G_p is the remaining game starting from the position p. Formally $G_p = \langle A_{1,p}, A_{2,p} \rangle$, where for each $i = 1, 2$, $A_{i,p}$ is defined by for all $(z_0, z_1, \cdots) \in Y^\omega$, $(z_0, z_1, \cdots) \in A_{i,p}$ if and only if $(y_0, y_1, \cdots, y_n, z_0, z_1, \cdots) \in A_i$, .

Remark 2.3.5. We have the convention that the player who moves first in a subgame G_p is the player who moves at p.

2.3.1 First Axiom

In order to rule out irrational behaviors such as player 1 choosing R in the game in Example 1, we propose a refined version of backward induction.

We said that L is a better choice than R because choosing L leads to secured win for player 1; while the win in the subgame following R is not secured in the sense that whether it leads to a win is completely determined by player 2 who is indifferent. So it is crucial for us to formally represent the difference between a secured and an unsecured win. The way we distinguish the secured and unsecured win is to regard a subgame in which a player is unable to secure a win as a subgame that he loses.

Let p be a position for player 1 to move. Let $G = \langle A_1, A_2 \rangle$ be such that, for each player i and each $y \in Y$, either $N_{p^\frown y} \subset A_i$ or $A_i \cap N_{p^\frown y} = \emptyset$. Note that this is equivalent to $A_{i,p^\frown y} = N_{p^\frown y}$ or $A_{i,p^\frown y} = \emptyset$ using the subgame notion $G_{p^\frown y} = \langle A_{1,p^\frown y}, A_{2,p^\frown y} \rangle$. I.e., $G_{p^\frown y}$ is a trivial game for each $y \in Y$.

Since it is player 1's turn to move at p, clearly, he can secure a win in the subgame G_p if and only if there exists some y such that if he chooses this y, he for sure wins, i.e., $\exists y (N_{p^\frown y} \subset A_1)$. While for player 2, he can secure a win for the subgame G_p if and only if the following is true:

1. Suppose player 1 has some choice y such that he wins the subgame $G_{p^\frown y}$, i.e., $N_{p^\frown y} \subset A_1$. So if there are more than one such y, player 1 could choose any one. In this case if player 2 want to guarantee a win for G_p he has to be sure that he wins $G_{p^\frown y}$ for all such y.

2. Otherwise, for all possible choice y, player 1 will lose the subgame $G_{p^\frown y}$, i.e., $N_{p^\frown y} \cap A_1 = \emptyset$. So player 1 is indifferent in choosing which y and any y is possible. In this case if player 2 want to guarantee a win for G_p he has to be sure that he wins $G_{p^\frown y}$ for all y.

Formally this can be summarized as the following formula

$$
\begin{aligned}
& [\exists y(N_{p^\frown y} \subset A_1) \rightarrow \forall y(N_{p^\frown y} \subset A_1 \rightarrow N_{p^\frown y} \subset A_2)] \\
\& \quad & [\forall y(N_{p^\frown y} \cap A_1 = \emptyset) \rightarrow \forall y(N_{p^\frown y} \subset A_2)].
\end{aligned} \tag{2.1}
$$

If condition (2.1) is violated then player 2 does not have a secured win for the subgame G_p.

We can now state part of the axiom.

Axiom 2.3.6 (Refined Backward Induction, Case 1)**.** Let G and p be as above. Then G can be reduced to $G^* = \langle A_1^*, A_2^* \rangle$ where each A_i^* is defined as follows.

1. $A_1^* = A_1 \cup N_p$ if $\exists y(N_{p^\frown y} \subset A_1)$ and $A_1^* = A_i$ otherwise.

2. $A_2^* = A_2 \cup N_p$ if condition (2.1) holds, and $A_2^* = A_2 \setminus N_p$ otherwise.

Remark 2.3.7. The implicit behavioral assumption behind item 1 simply says that in case that player has some choice y at position p which makes him win the subgame G_p, then he *should* choose one of such ys to win the game. But there is no further requirement as which one he should choose.

Item 2 indicates, as we have shown above, when G_p is a secured win for the inactive player at p.

Now we consider a slightly different situation. Suppose that there is a y such that player 1 is sure to win the subgame $G_{p^\frown y}$ and player 2 is sure to lose this subgame, i.e.

$$N_{p^\frown y} \subset A_1 \text{ and } N_{p^\frown y} \cap A_2 = \emptyset. \tag{2.2}$$

By a similar analysis as above, we can conclude that player 1 will win the game G_p and "player 2 will lose" since he is not able to secure a win if player 1 chooses this particular y. Note that in this situation we do not even need to know what are the outcomes in those subgames $G_{p^\frown y}$ for other ys. Formally,

Axiom 2.3.8 (Refined Backward Induction, Case 2)**.** Let G and p be as above. Then $G = \langle A_1, A_2 \rangle$ can be reduced to $\langle A_1 \cup N_p, A_2 \setminus N_p \rangle$.

2.3.2 Second Axiom

The way we solve the problem in Example 2 is to introduce a behavioral axiom requiring the players to cooperate when they have the same payoff sets. For instance, it has to force the players to say "continue" at each stage in this example. Formally we would require that any game $G = \langle A_1, A_2 \rangle$ such that $A_1 = A_2 \neq \emptyset$ can be reduced to a trivial game $G^* = \langle Y^\omega, Y^\omega \rangle$. This is the main idea of the axiom.

But it is necessary for us to relax the condition $A_1 = A_2 \neq \emptyset$ slightly while still capturing the same idea. For instance, consider the following variant of Example 2.

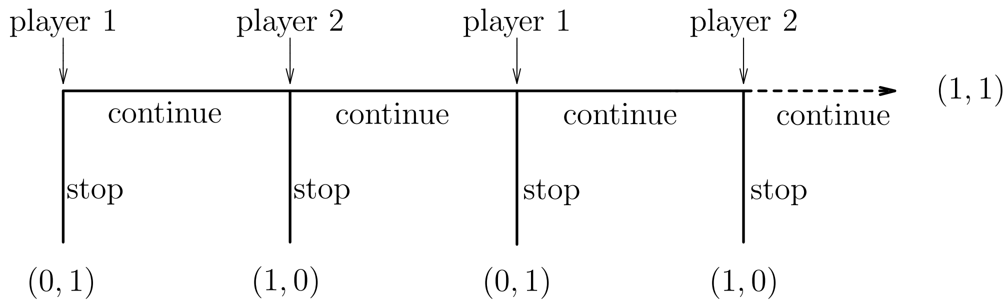

FIGURE 3

Although the players have different payoff sets in this game, the same analysis showing that the rational outcome should be that the players cooperate so that they both win the game still applies.

If a player says "stop" at any stage, *he* is sure to lose the game. So suppose they both avoid the sure-to-lose moves, then they will win the game since then they can only say "continue" forever. So in this situation rational players should also cooperate.

Putting the intuitions of these two examples together we have the following more general behavioral axiom: If, modulo the effect of avoiding the sure-to-lose moves on the payoff sets, they have essentially the same payoff set, then they should cooperate to win the game.

Formally, we say that player i is sure to lose after a choice y at position p if $N_{p^\frown y} \cap A_i = \emptyset$, or equivalently, $A_{i,p^\frown y} = \emptyset$. So the effect on the payoff sets A_i by avoiding all the sure-to-moves can be represented by a set

$$K = \{N_{p^\frown y} \mid (1)\ \exists i (i \text{ moves at } q)\ \&\ (2)\ G_{p^\frown y} \text{ is trivial}$$

$$\&\ (3)\ G_p \text{ is not trivial}\ \&\ (4)\ A_{i,p^\frown y} = \emptyset\}.$$

Define, for each $i = 1, 2$, $\tilde{A}_i = A_i \setminus K$. Then A_i is the new payoff set of player i if both players avoids sure-to-lose moves.

The axiom states that if

$$\tilde{A}_1 = \tilde{A}_2 \neq \emptyset, \tag{2.3}$$

i.e., the players have the same payoff functions modulo the irrational moves, then they should cooperate to win the game. Putting it in our reduction language, we have

Axiom 2.3.9 (Cooperation Axiom)**.** Suppose (4.1) holds, then G can be reduced to the trivial game $\langle Y^\omega, Y^\omega \rangle$.

2.3.3 Third Axiom

Definition 2.3.10. Let $M_1 \subset Y^{<\omega}$ be the collection of all positions for player 1 to move. A strategy σ for player 1 is a function from M_1 to Y.

Let σ and τ be strategies for player 1 and 2 respectively. A play according to σ and τ is the sequence

$$\sigma * \tau = \big(\sigma(\emptyset), \tau(\sigma(\emptyset)), \sigma((\sigma(\emptyset), \tau(\sigma(\emptyset))), \cdots\big).$$

Definition 2.3.11. A strategy σ for player 1 is a winning strategy in the game $G = \langle A_1, A_2 \rangle$ if and only if for each strategy τ of player 2, $\sigma * \tau \in A_1$.

Axiom 2.3.12.

1. Suppose player 1 has a winning strategy in $G = \langle A_1, Y^\omega \rangle$, then G can be reduced to $\langle Y^\omega, Y^\omega \rangle$.

2. Suppose player 1 has a winning strategy in $G = \langle A_1, \emptyset \rangle$, then G can be reduced to $\langle Y^\omega, \emptyset \rangle$.

2.4 Determinacy of finite games

This section illustrates the working of determinacy by proving that all finite games are determined. The following definition gives the usual notion of finite games in the infinite setting.

Definition 2.4.1. A game $G = \langle A_1, A_2 \rangle$ is called finite if there exists a natural number n such that for each sequence $p \in Y^{<\omega}$ of length n, either $N_p \subset A_i$ or $N_p \cap A_i = \emptyset$, for each $i = 1, 2$. I.e., for each such p, G_p is a trivial game.

Like the usual backward induction, the refined backward induction alone is able to solve for any finite games.

Theorem 2.4.2. *Finite games are determined.*

Proof. For simplicity, consider the case Y is a finite set. The general case is left to Section **??**. Let G be a finite game of length n on Y, i.e., G_p is trivial for each p of length n.

Define a reduction chain $\langle G_k \mid k \leq m \rangle$ (m to be determined), by induction on k. Let $G_0 = G$. Suppose G_{k-1} has been defined. Pick an element $p \in Y^{<\omega}$ such that $G_{k-1,p}$ is not trivial and for each $y \in Y$, $G_{k-1,p^\frown y}$ is trivial. So G_{k-1} and p satisfy the conditions for the refined backward induction. Let $G_k = G^*_{k-1}$, where G^*_{k-1} is obtained by applying the axiom to G_{k-1}. If there is no such p, then G_{k-1} is already a trivial game, let $m = k - 1$. By finiteness of Y, the reduction process terminates at some finite stages. So m exists. Thus we obtain a finite reduction chain for G and G is determined. □

2.5 Discussion

We now turn to a discussion of determinacy and the axioms.

2.5.1 Interpretation of determinacy

The outcome given by determinacy requires a different interpretation. The reason is that we use the expression "a player lose a game" not only its literal meaning, but also as a device to record an unsecured win.

Suppose G reduces to a trivial game $G^* = \langle A_1^*, A_2^* \rangle$. If player i wins the game G, i.e., $A_i^* = Y^\omega$, then indeed player i can win the game if he (or both players) plays rationally. But if player i loses the game G, i.e., $A_i^* = \emptyset$, then there are two possibilities.

The first possibility is that indeed player i is for sure to lose the game G. The second possibility is that at certain stage of the reduction process the refined backward induction is applied, and "player i loses the game G" only means that he has an unsecured win at certain subgame. So in this case it is indeed possible for player player i to lose in the game G by losing that subgame, but it is also possible for him to win G. Which case happens will depend entirely on the other player' choice and, moreover, he is indifferent in which one to choose.

Summing up, "player i wins the game G", i.e., $A_i^* = Y^\omega$, indeed means that player i can win the game but "player i loses the game G", i.e., $A_i^* = \emptyset$, does not necessarily implies that he will for sure lose the game.

2.5.2 Comparison with the usual backward induction

To see exactly how the refined backward induction differs from the usual backward induction let's define a notation $v_i(G)$, where $v_i(G) = 1$ if player i wins G, otherwise $v_i(G) = 0$. Let p be a position for player 1 to move. The refined backward induction can be interpreted as the following two equations,

$$v_1(G_p) = \max\{v_1(G_{p^\frown y}) \mid y \in Y\}, \tag{2.4}$$

$$v_2(G_p) = \min\{v_2(G_{p^\frown y}) \mid y \in Y \ \& \ v_1(G_{p^\frown y}) = v_1(G_p)\}. \tag{2.5}$$

The usual backward induction also has equation 2.4, but equation 2.5 is weakened to

$$v_2(G_p) \in \{v_2(G_{p^\frown y}) \mid y \in Y \ \& \ v_1(G_{p^\frown y}) = v_1(G_p)\}. \tag{2.6}$$

Hence the usual backward induction allows for the value $v_2(G_p)$ to be that of arbitrary subgame $G_{p^\frown y}$ as long as it is possible for player 1 to choose y. But refined backward induction requires that 2 has to guarantee that he can at least have $v_2(G_p)$, regardless of what player 1 will choose as long as player 1 is rational and chooses one that gives himself highest possible payoff.

Equation (2.5) and (2.6) coincide in case that G is in general position, so there is no difference between refined backward induction and the usual backward induction in this case. But in our context, the games that are in general position, i.e. for each i there is no more than one y such that $A_{i,p^\frown y} = N_{p^\frown y}$ and no more than one y such that $A_{i,p^\frown y} = \emptyset$, are actually not general.

2.5.3 Comparison with weak dominance

Our second axiom, and some case of the third axioms, can be derived from weak dominance. But we are still using these axioms rather than weak dominance as the basic principles for several reasons.

Firstly they are weaker than weak dominance since weak dominance implies them but certainly they do not imply weak dominance. In setting axiom systems we always choose the weakest possible ones.

Secondly they are concepts in quite different settings. Weak dominance is a concept for normal form games and the axioms are designed for applications in dynamic games with perfect information. In applying weak dominance to them we are implicitly reducing the dynamic games to one-shot games, losing the important dynamic character.

Thirdly, the axioms are local, or atomic, statements in the sense that they apply only to some very special games satisfying certain conditions. But we can try weak dominance for any games. So weak dominance is a global statement. In setting axioms, it would be desirable to minimize the domain of application.

Fourthly, and more seriously, when applied to these games weak dominance is inconsistent with our second axiom. Consider the example in Figure 4.

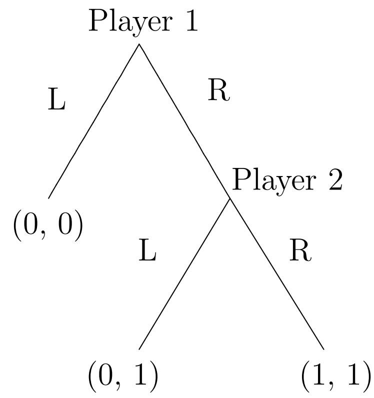

FIGURE 4

Weak dominance suggest player 1 choosing R in the first period and the payoff of player 2 in the whole game is 1. Following the same analysis that leads to the refined backward induction, it seems too strong to come to this conclusion: It is possible for player 2 in the second period to choose L, hence it is possible for player 1 to lose by choosing R. Basing on this, player 1 can choose L in the first period so that the payoff of player 2 is 0.

Formally, our second axiom does not imply that the payoff of player 2 in this game must for sure be 1. This contradicts weak dominance.

2.5.4 Comparison with subgame perfect Nash equilibrium

We have already seen that each axiom is actually a unique refinement of subgame perfect Nash equilibria in some special games. I.e, the axioms refine subgame perfect Nash equilibrium at the local level. The following theorem shows the outcome of a determined game is supported by a subgame perfect Nash equilibrium. Namely, determinacy is a refinement of subgame perfect Nash equilibrium when looked globally.

Definition 2.5.1. G is called strictly determined if G_p is determined for each $p \in Y^{<\omega}$.

Theorem 2.5.2. *Let $G = \langle A_1, A_2 \rangle$ be strictly determined. Then there exist strategies σ and τ, for player 1 and 2 respectively, such that*

1. *(σ, τ) is a subgame perfect Nash equilibrium and*

2. *$\sigma * \tau \in A_i$ if and only if player i wins the game, $i = 1, 2$.*

Remark 2.5.3. An implication of determinacy for the refinement literature is the following. In refining solution concepts we should focus on the uniqueness of outcome rather than on the strategy profiles. For example, in a trivial game any strategy is as good as the others, either in the sense of subgame perfect Nash equilibrium or in the sense of any possible future refinements. But the outcome of the game, whether a player is going to win or lose the game, is uniquely determined. As long as different sets of strategies give the same result as suggested by determinacy, it seems that there is no point to further distinguish them.

2.5.5 Conclusion

By iterating three simple behavioral axioms we build a new solution concept for the class of two-person perfect information games with characteristic payoff functions. These axioms are shown consistent, complete and independent. Moreover, it determines a unique vector of payoffs that corresponds to a subgame perfect Nash equilibrium. Thus it can be viewed as a unique refinement of subgame perfect Nash equilibria.

2.6 Proofs

2.6.1 Ordinals and the complete version of determinacy

The ordinal numbers were Georg Cantor's deepest contribution to mathematics. After the natural numbers 0, 1, ... n, ... comes the first infinite ordinal number ω, followed by $\omega+1$, $\omega+2$, ... , $\omega+\omega$, . . . and so forth. ω is the first limit ordinal as it is neither 0 nor a successor ordinal. We follow the von Neumann convention, according to which each ordinal number α is identified with the set $\{\nu|\nu < \alpha\}$ of its predecessors. The $\in$ relation on ordinals thus coincides with $<$. We have $0 = \emptyset$ and $\alpha + 1 = \alpha \cup \{\alpha\}$. In particular, $\omega = \{0, 1, 2, \cdots\}$ is the set of all natural numbers. Thus we arrive at the following picture

$$0, 1, \cdots, n, \cdots, \omega, \omega+1, \omega+2, \cdots, \omega+\omega, \cdots, \omega^2, \cdots.$$

The version of determinacy we gave in Definition 2.2.6 is inadequate since there are games in which we need more than finite steps of reduction in order to have a trivial game. For instance, in Theorem 2.4.2, if the choice set Y is infinite, say $Y = \omega$, then Definition 2.2.6 does not work since, no matter how long the reduction chain is, a finite chain is not able to solve an infinite number of subgames $\{G_{p^\frown y}|y \in \omega\}$. For instance, we need to apply the refined backward induction axiom for each subgame $G_{p^\frown n}$, $n \in \omega$. Then at least we need an infinite chain $\langle G_0, G_1, G_2, \cdots\rangle$ each reduction $G_k \to G_{k+1}$ solving the subgame $G_{p^\frown k}$. So we need to allow for arbitrarily infinitely long reduction chains in the definition of determinacy. The notion of ordinals is a nice tool for this purpose. In the following, Greek letters $\alpha, \beta, \gamma, \delta$ denote general ordinals numbers, θ denotes a limit ordinal.

We shall use intensively the techniques of definition and proof by induction on ordinals. They are natural extensions of definition and proof by induction on natural numbers since ordinals are just extensions of natural numbers.

For example, if we need to define a sequence of subsets of Y^ω, $\langle A_\alpha \mid \alpha < \gamma \rangle$, it suffices to proceed as follows. First define what A_0 is. Given that α is defined, proceed to define $\alpha + 1$. If $\theta < \gamma$ is a limit ordinal and given that A_α is defined for each $\alpha < \theta$, describe how A_θ is defined.

Suppose we are to prove a proposition $P(\alpha)$ with an ordinal α is involved. We proceed similarly by proving the base case $P(0)$; given the induction hypothesis $P(\alpha)$, prove $P(\alpha + 1)$; and given the induction hypothesis $P(\alpha)$ for each $\alpha < \theta$, where θ is a limit ordinal, prove $P(\theta)$.

Definition 2.6.1. Let θ be a limit ordinal and let $\langle A_\alpha \mid \alpha < \theta \rangle$ be a sequence of subsets of Y^ω. Then $A_\theta = \lim\limits_{\alpha < \theta} A_\alpha$ is defined by letting, for all $f \in Y^\omega$, $f \in A_\theta$ if and only if there exists $\beta < \theta$ such that $f \in A_\alpha$ for all $\alpha > \beta$. I.e., an element f is in the limit A_θ if and only if it is in A_α for all but a bounded initial segment of $\langle A_\alpha \mid \alpha < \theta \rangle$.

Definition 2.6.2. A reduction chain is a sequence $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$ of games such that each $G_{\alpha+1}$ is obtained from G_α by applications of the behavioral axioms defined in section 2.3 and, if $\theta \leqslant \gamma$ is a limit ordinal, $G_\theta = \langle \lim\limits_{\alpha<\theta} A_{1,\alpha}, \lim\limits_{\alpha<\theta} A_{2,\alpha} \rangle$

The following is the complete version of the definition of determinacy.

Definition 2.6.3. A game $G = \langle A_1, A_2 \rangle$ is called determined if there exists a reduction chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$ such that $G_0 = G$ and G_γ is trivial.

The following are the corresponding versions for Theorem 2.6.33 and Definition 2.2.10.

Theorem 2.6.4. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$ and $\langle H_\alpha = \langle B_{1,\alpha}, B_{2,\alpha} \rangle \mid \alpha \leqslant \delta \rangle$ be two reduction chains for the game $G = \langle A_1, A_2 \rangle$ such that $G_0 = H_0 = G$ and G_γ, H_δ are trivial. Then $G_\gamma = H_\delta$.*

Definition 2.6.5. Let $G = \langle A_1, A_2 \rangle$ be a determined game with a reduction chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$. We say that player i wins the game G if he wins G_γ, otherwise he lose G.

The last theorem says that the axioms are independent.

Theorem 2.6.6. *If any of the following items are dropped then there will be a closed game that is not determined:*

Case 1 of refined backward induction; Case 2 of refined backward induction;

The cooperation axiom; Case 1 of the third axiom; Case 2 of the third axiom.

2.6.2 Preliminaries

Before proceeding we need to establish some preliminary results in this section.

Notation 2.6.7. Let $f = (y_0, y_1, \cdots) \in Y^\omega$ and $k \geq 1$, $f \upharpoonright k$ denotes $(y_0, y_1, \cdots, y_{k-1})$. Similarly, let $p = (y_0, y_1, \cdots, y_n) \in Y^{<\omega}$ and $k \leq n+1$, then $p \upharpoonright k$ denotes $(y_0, y_1, \cdots, y_{k-1})$.

Definition 2.6.8. Let $f, f_n \in Y^\omega$, $n \geq 0$. Say that the sequence f_n converges to f, written as $f_n \to f$ or $\lim_{n\to\infty} f_n = f$, if there exist a nondecreasing sequence of natural numbers $\langle k_n \mid n \geq 0\rangle$ such that $k_n \to \infty$ and $f_n \restriction k_n = f \restriction k_n$ for each n.

Lemma 2.6.9. *Let A be a subset of Y^ω. A is closed if and only if for each sequence $\langle f_n \mid n \geq 0\rangle$ such that for each n, $f_n \in A$ and $f_n \to f$, then $f \in A$.*

Proof. ($\Longrightarrow$) Assume, towards a contradiction, $f \in Y^\omega \setminus A$. Since $Y^\omega \setminus A$ is open, by definition, it's a union of basic open neighborhoods N_p for some $p \in Y^{<\omega}$. So there is a particular p such that $f \in N_p$. Let k be the length of p, then $f \restriction k = p$. So for all $n > n_0$, where n_0 is the least integer such that $k_{n_0} > k$, $f_n \restriction k = f \restriction k = p$. It follows that $f_n \in N_p \subset (Y^\omega \setminus A)$ for all $n > n_0$, contradicting the fact that $f_n \in A$ for all n. Therefore $f \in A$.

($\Longleftarrow$) It suffices to show that $Y^\omega \setminus A$ is open, i.e., $Y^\omega \setminus A$ is a union of basic open neighborhoods of the form N_p. That is, each $f \in Y^\omega \setminus A$ sits in $N_{f\restriction k} \subset Y^\omega \setminus A$ for some k. Take an arbitrary $f \in Y^\omega \setminus A$. Suppose, towards a contradiction, that for all $k \geq 0$, $N_{f\restriction k} \not\subseteq Y^\omega \setminus A$. Then, for each k, there is a $g_k \in N_{f\restriction k}$ and $g_k \notin Y^\omega \setminus A$. Equivalently, for each k, $g \restriction k = f \restriction k$ and $g_k \in A$. So $g_k \to f$. By assumption, $f \in A$, a contradiction. So $Y^\omega \setminus A$ is open and A is closed. $\square$

Lemma 2.6.10. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha < \theta\rangle$ be a reduction chain such that θ is a limit ordinal. Then for each $f \in Y^\omega$ and each i there exists $\beta < \theta$ such that either $f \in A_{i,\alpha}$ for all $\alpha > \beta$ or $f \notin A_{i,\alpha}$ for all $\alpha > \beta$.*

Proof. Fix i and an f. Since f will be removed from or added to the payoff set of i only if an axiom is applied to a subgame $G_{f\restriction k}$ for some natural number $k \geq 0$. By this can happen only with a finite sequence of $k_1 > k_2 > \cdots > k_n \geq 0$. $\square$

Lemma 2.6.11. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leq \gamma\rangle$ be a reduction chain such that G_0 is a closed game, then each G_α, $\alpha \leq \gamma$, is also a closed game.*

Proof. Fix i. We proceed by induction on $\alpha \leq \gamma$ to show each $A_{i,\alpha}$ is closed.

$A_{i,0}$ is closed by assumption.

Suppose $A_{i,\alpha}$ is closed. $A_{i,\alpha+1}$ is either $A_{i,\alpha}$, $A_{i,\alpha} \cup N_p$ or $A_{i,\alpha} \setminus N_p$ for some p, so $A_{i,\alpha+1}$ is closed.

Let θ be a limit ordinal and each $A_{i,\alpha}$, $\alpha < \theta$, is closed. Suppose that $f_n \in A_{i,\theta} = \lim_{\alpha<\theta} A_{i,\alpha}$ for each $n = 1, 2, \cdots$, and $\lim_{n\to\infty} f_n = f$, we want to show that $f \in A_{i,\theta}$.

From the definition of $\lim_{\alpha<\theta} A_{i,\theta}$, let β_n be the least ordinal such that $f_n \in A_{i,\alpha}$ for each $\alpha > \beta_n$. By choosing a subsequence if necessary, we may assume that $\beta_1 \leq \beta_2 \leq \cdots$ and $f_n \restriction n = f \restriction n$.

Either there exists some n_0 such that $\beta_{n_0} = \beta_{n_0+1} = \beta_{n_0+2} = \cdots$ or there is a subsequence $\langle \beta_{n_k} \mid k \geq 1\rangle$ of $\langle \beta_n \mid n \geq 1\rangle$ such that $\beta_{n_1} < \beta_{n_2} < \beta_{n_3} < \cdots$.

Case 1. $f_n \in A_{i,\alpha}$ for all $n \geq 0$ and all $\alpha > \beta_0$. By the induction hypothesis, i.e., each $A_{i,\alpha}$, $\alpha < \theta$, is closed, we have $f \in A_{i,\alpha}$ for each $\alpha > \beta_0$. So $f \in A_{i,\theta}$.

Case 2. Without lost of generality, let $\beta_1 < \beta_2 < \beta_3 < \cdots$.

We first show that $f \in A_{i,0}$. For each n, if $f_n \in A_{i,0}$, let $g_n = f_n$. Otherwise f_n is added to A_{i,β_n+1} by letting $A_{i,\beta_n+1} = A_{i,\beta_n} \cup N_{f_n \restriction k_n}$ for some k_n. Recall that this is the case if and only if i wins the subgame $G_{\beta_n, f_n \restriction k_n}$. In particular, initially $A_{0,i,f_n \restriction k_n}$ cannot be empty. Pick a $g_n \in A_{0,i} \cap N_{f_n \restriction k_n}$. Again by choosing a subsequence if necessary let $k_1 \leq k_2 \leq \cdots$ and $k_n \geq n$ for each n. Then each $g_n \in A_{0,i}$ and $\lim_{n\to\infty} g_n = \lim_{n\to\infty} f_n = f$. By closedness of $A_{0,i}$, $f \in A_{0,i}$.

We next show that $f \in A_{i,\alpha}$ for all but a bounded initial segment of $\langle 0, 1, \cdots, \theta\rangle$. Otherwise, since the status of whether or not $f \in A_{i,\alpha}$ is going to change at most finitely many times by Lemma 2.6.10, let β be the last ordinal such that $f \in A_{i,\beta}$. Such β exists since $f \in A_{i,0}$. Then by definition of $A_{i,\beta+1}$ we must have that $A_{i,\beta+1}$ is obtained by $A_{i,\beta+1} = A_{i,\beta} \setminus N_{f\restriction l}$ for some l. This contradicts the assumption that i wins the subgame $G_{f_n\restriction k_n}$ for infinitely many $n > l$. So $f \in A_{i,\alpha}$ for all but a bounded initial segment of $\alpha < \theta$, hence $f \in A_{i,\theta}$. So $A_{i,\theta}$ is closed.

This completes the induction. $\square$

Lemma 2.6.12. *For each G there exists an ordinal γ such that there is no reduction chain for G of length greater than γ. Formally let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leq \delta\rangle$ be a reduction chain for G such that for each $\alpha \leq \delta$, $G_\alpha \neq G_{\alpha+1}$ (i.e., no trivial application of the axioms). Then $\delta < \gamma$.*

Remark 2.6.13. Lemma 2.6.12 implies that if we can define a reduction chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leq \delta\rangle$ such that for each α, if G_α is not trivial then $G_\alpha \neq G_{\alpha+1}$. Then at certain stage γ, G_γ must be trivial. That is, a reduction chain cannot be arbitrarily long and must terminate at some stage.

Proof of Lemma 2.6.12. Bounded by the cardinality of $Y^{<\omega}$. $\square$

Definition 2.6.14. A set $T \subset Y^{<\omega}$ is called a tree if $(y_0, y_1, \cdots, y_m) \in T$, $m \geq 1$, implies $(y_0, y_1, \cdots, y_{m-1}) \in T$.

Definition 2.6.15. A tree T is called well-founded if for all $f \in Y^\omega$ there exists k such that $(f \restriction k) \notin T$.

Definition 2.6.16. Associate to each closed set $A \subset Y^\omega$ a tree T as follows: $p \in T$ if and only if there exists $f \in A$ and $k \geq 0$ such that $p = f \restriction k$. T is called the tree representation of A.

Remark 2.6.17. By Lemma 2.6.9, if $f \restriction k \in T$ for each $k \geq 0$, then indeed $f \in A$.

Definition 2.6.18. A t-strategy σ for player 1 is the smallest subset of $Y^{<\omega}$ such that,

1. $\emptyset \in \sigma$.

2. If $p = (y_0, y_1, \cdots, y_n) \in \sigma$ and p is a position for player 1 to move, then there exists a unique $y \in Y$ such that $p^\frown y \in \sigma$.

3. If $p = (y_0, y_1, \cdots, y_n) \in \sigma$ and p is not a position for player 1 to move, then, for all $y \in Y$, $p^\frown y \in \sigma$.

Definition 2.6.19. A t-strategy σ for player 1 is a winning t-strategy in the game $G = \langle A_1, A_2 \rangle$ if and only if for all $f \in Y^\omega$, $(f \restriction n) \in \sigma$ for each n implies $f \in A_1$.

Lemma 2.6.20. *Player 1 has a winning strategy in a game G if and only if he has a winning t-strategy.*

Proof. clear. $\square$

The following lemma complements Axiom 2.3.12.

Lemma 2.6.21. *Suppose player 1 does not have a winning strategy in $G = \langle A_1, Y^\omega \rangle$, where A_1 is a closed set, then G can be reduced to $\langle \emptyset, Y^\omega \rangle$.*

Suppose player 1 does not have a winning strategy in $G = \langle A_1, \emptyset \rangle$, where A_1 is a closed set, then G can be reduced to $\langle \emptyset, \emptyset \rangle$.

Proof. We shall only show the first case, the proof for the second case is similar. Define a reduction chain $\langle G_\alpha = \langle B_\alpha, Y^\omega \rangle \mid \alpha \leq \gamma \rangle$ such that if player 1 has a winning strategy in G_α for some α, then it has a winning strategy for G.

Let $B_0 = A_1$. Suppose that B_α is defined and player 1 does not have a winning strategy in G_α. If $B_\alpha = T_\alpha = \emptyset$, then let $\gamma = \alpha$ and we are done.

Otherwise, by Lemma 2.6.11, B_α is closed and nonempty. Let T_α be the tree representing B_α. By the assumption that 1 does not have a winning strategy and Lemma 2.6.20, T_α dose not contain a t-strategy.

Hence, by the definition of a t-strategy (Definition 2.6.18), there are two possibilities.

1. there exists some $p \in T_\alpha$ of even length k, i.e., p is a position for player 1 to move, such that for all $y \in Y$, $(p^\frown y) \notin T_\alpha$. That is, $N_{p^\frown y} \cap B_\alpha = \emptyset$ for each $y \in Y$.

2. there exists some $p \in T_\alpha$ of odd length k, i.e., p is a position for player 2 to move, such that for some $y \in Y$, $(p^\frown y) \notin T_\alpha$. That is, $N_{p^\frown y} \cap B_\alpha = \emptyset$ for this y.

In either case, the refined backward induction applies. Pick one such p and let $B_{\alpha+1} = B_\alpha \setminus N_{p \restriction (k-1)}$.

If θ is a limit, let $B_\theta = \lim\limits_{\alpha<\theta} B_\alpha$.

Since $B_\alpha \supsetneq B_{\alpha+1}$ and player does not have a winning strategy in G_α, player 1 cannot have a winning strategy for $G_{\alpha+1}$. Similarly player 1 does not have a winning strategy in G_θ for θ a limit ordinal.

So $\langle G_\alpha = \langle B_\alpha, Y^\omega \rangle \mid \alpha \leq \gamma \rangle$ defines a reduction chain with $G_0 = G$ and $G_\gamma = \langle \emptyset, Y^\omega \rangle$ and the lemma is proved. □

Combining Axiom 2.3.12 and Lemma 2.6.21 we have

Lemma 2.6.22. *Suppose A_1 is a closed set then $G = \langle A_1, Y^\omega \rangle$ can be reduced to $\langle A_1^*, Y^\omega \rangle$, where $A_1^* = Y^\omega$ or $\emptyset$ according to whether or not player 1 has a winning strategy.*

Suppose A_1 is a closed set then $G = \langle A_1, \emptyset \rangle$ can be reduced to $\langle A_1^*, \emptyset \rangle$, where $A_1^* = Y^\omega$ or $\emptyset$ according to whether or not player 1 has a winning strategy.

The following lemma is also useful in the following discussion.

Lemma 2.6.23. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leq \gamma \rangle$ be a sequence of games such that*

1. *For each $\alpha < \gamma$ there exists a reduction chain $\langle H_{\beta,\gamma_\alpha} = \langle B_{1,\beta,\gamma_\alpha}, B_{2,\beta,\gamma_\alpha} \rangle \mid \beta \leq \gamma_\alpha \rangle$ such that $H_{0,\gamma_\alpha} = G_\alpha$, $H_{\gamma_\alpha,\gamma_\alpha} = G_{\alpha+1}$;*

2. *For each $\theta \leq \gamma$ a limit ordinal, $G_\theta = \langle \lim\limits_{\alpha<\theta} A_{1,\alpha}, \lim\limits_{\alpha<\theta} A_{2,\alpha} \rangle$.*

Then there exists a reduction chain starting with G_0 and ending with G_γ.

Proof. Order the set of pairs $P = \{\langle \beta, \gamma_\alpha \rangle \mid \beta \leq \gamma_\alpha \leq \gamma\}$ by $\langle \beta_1, \gamma_{\alpha_1} \rangle < \langle \beta_2, \gamma_{\alpha_2} \rangle$ if $\gamma_{\alpha_1} < \gamma_{\alpha_2}$ or, $\gamma_{\alpha_1} = \gamma_{\alpha_2}$ and $\beta_1 < \beta_2$. It is easy to see that $<$ is a well ordering of P. Let δ be the order type of P and $\pi : P \to \delta$ be the canonical order isomorphism. Let $I_\xi = H_{\pi^{-1}(\xi)}$, then the sequence $\langle I_\xi \mid \xi < \delta \rangle$ forms a reduction chain as required. □

2.6.3 Proof of Theorem 2.2.12

Proof. We proceed in two steps.

Step 1. Define a sequence of games $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leq \gamma\rangle$, γ to be determined, inductively.

Let $A_{i,0} = A_i$ for each i.

Suppose that $A_{i,\alpha}$ has been defined, we proceed to define $A_{i,\alpha+1}$ for each i.

For any $f \in (A_{1,\alpha} \setminus A_{2,\alpha}) \cup (A_{2,\alpha} \setminus A_{1,\alpha})$, say $f \in A_{1,\alpha}$ and $f \notin A_{2,\alpha}$. By closedness of $A_{2,\alpha}$ there exists a least k such that $N_{f\restriction k} \cap A_{2,\alpha} = \emptyset$, or, $A_{2,\alpha,f\restriction k} = \emptyset$. Consider the subgame of G_α starting at $f \restriction k$, $G_{\alpha,f\restriction k}$. Since player 2 has empty payoff set in this subgame, by Lemma 2.6.22 we know that this subgame is determined. Let l be the player who moves at the position $f \restriction (k-1)$. Then either l wins $G_{\alpha,f\restriction k}$, which will be called case 1, or l loses $G_{\alpha,f\restriction k}$, called case 2.

If there does not exist f satisfying the condition of case 1, we are done. Let $\gamma = \alpha$.

Otherwise pick a pair $\langle f, k\rangle$ such that

1. f and k satisfy the condition of case 1. I.e., say, $f \in A_{1,\alpha}$ and $N_{f\restriction k} \cap A_{2,\alpha} = \emptyset$ and l wins $G_{\alpha,f\restriction(k-1)}$; l is the player that moves at $f \restriction (k-1)$ and $(3-l)$ is the inactive player.

2. k is smallest possible among all such pairs $\langle f', k'\rangle$.

Let $A_{l,\alpha+1} = A_{l,\alpha} \cup N_{f\restriction(k-1)}$ and $A_{(3-l),\alpha+1} = A_{(3-l),\alpha} \setminus N_{f\restriction(k-1)}$. (If l wins $G_{\alpha,f\restriction k}$, so it is not the case that $A_{l,\alpha} \cap N_{f\restriction k} = \emptyset$. Then, by the definition of f, it must be that $A_{(3-l),\alpha} \cap N_{f\restriction k} = \emptyset$. Hence $(3-l)$ must lose the game $G_{f\restriction k}$. So $A_{(3-l),\alpha+1} = A_{(3-l),\alpha} \setminus N_{f\restriction(k-1)}$ according to case 2 of refined backward induction.) According to Lemma 2.6.22 and case 2 of refined backward induction, there exists a reduction chain linking G_α and $G_{\alpha+1}$.

If θ is a limit ordinal, let $A_{i,\theta} = \lim_{\alpha<\theta} A_{i,\alpha}$.

Clearly the procedure terminates at certain stage and γ exists. By Lemma 2.6.23, there is a reduction chain linking G and G_γ.

Step 2. Show that G_γ is determined.

Since f of case 1 does not exist for G_γ, $\tilde{A}_{1,\gamma} = \tilde{A}_{2,\gamma} = A_{1,\gamma} \cap A_{2,\gamma}$. Here each $\tilde{A}_{i,\gamma}$ is defined as in equation 4.1. There are two possibilities.

If $A_{1,\gamma} \cap A_{2,\gamma} \neq \emptyset$, by the cooperation axiom, G_γ is determined.

Now assume that $A_{1,\gamma} \cap A_{2,\gamma} = \emptyset$, we will show that G_γ is determined.

Let T be the collection of all those finite sequences of the form $f \restriction m$, $m \leqslant k$, where f and k are those defined in case 2, with no other $k' < k$ such that f, k' also satisfies that.

Let T_0 be the collection of all those pairs $\langle f, k\rangle$ such that

1. f and k satisfy the condition of case 2 of $G_{f\restriction k}$. I.e., say, $f \in A_{1,\alpha}$ and $N_{f\restriction k} \cap A_{2,\alpha} = \emptyset$ and l loses $G_{\alpha, f\restriction(k-1)}$; l is the player that moves at $f \restriction (k-1)$ and $(3-l)$ is the inactive player.

2. k is smallest possible among all such pairs $\langle f', k'\rangle$.

Let $T = \{p \mid \exists\langle f, k\rangle \in T_0, \exists m \leq k \text{ such that } p = f \restriction m\}$.

Claim 2.6.24. *T is well-founded.*

An induction on the well-founded tree T with each induction step applying case 1 of refined backward induction will give us the determinacy of G_γ.

Formally, we will define a reduction chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \gamma \leq \alpha \leq \delta\rangle$ as follows.

Let G_β, $\beta \geq \gamma$, be defined. Search for a $p \in T$ such that for all $y \in Y$, $G_{\beta,p^\frown y}$ is trivial but not $G_{\beta,p}$. Apply refined backward induction to G_β to get $G_{\beta+1}$.

Claim 2.6.25. *If G_β is not trivial, such p exists.*

If $\theta > \gamma$ is a limit ordinal, let $G_\theta = \lim\limits_{\beta<\theta} G_\beta$.

Let δ be the least ordinal such that G_δ is trivial. So the chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \gamma \leq \alpha \leq \delta\rangle$ a reduction chain for G_γ and G_γ is determined. □

Proof of Claim 2.6.24. Suppose not, then there exists f and a strictly increasing sequence of natural numbers $\langle k_n \mid n \geq 0\rangle$ such that for all n, $(f \upharpoonright k_n) \in T$.

We shall show that $f \in A_{i,\gamma}$ for each $i = 1, 2$, contradicting our assumption that $A_{1,\gamma} \cap A_{2,\gamma} = \emptyset$.

First suppose $f \notin A_{i,\gamma}$ for each $i = 1, 2$. By closedness of each $A_{i,\gamma}$, there exists k such that $N_{f\upharpoonright k} \cap A_{i,\gamma} = \emptyset$ for each i. Take an $k_n > k$. By definition of $(f \upharpoonright k_n)$ being in T, there must be a g and an $s \geq k_n$ such that $g \upharpoonright k_n = f \upharpoonright k_n$ and $\langle g, s\rangle \in T_0$. In particular, $g \in A_{i,\gamma}$ for some i. So $g \in A_{i,\gamma} \cap N_{f\upharpoonright k}$, a contradiction. So $f \in A_{i,\gamma}$ for at least one i.

Say, $f \in A_{1,\gamma}$. Suppose that $f \notin A_{2,\gamma}$. Again by closedness of $A_{2,\gamma}$, choose a least k such that $N_{f\upharpoonright k} \cap A_{2,\gamma} = \emptyset$. By the definition of G_γ, $f \upharpoonright (k-1)$ is a position for a player that loses $G_{f\upharpoonright k}$ to move. For $f \upharpoonright k_n$ to be in T, there must be a g and an $s \geq k_n > k$ such that $g \upharpoonright k_n = f \upharpoonright k_n$ and $\langle g, s\rangle \in T_0$. But this is impossible since $\langle g, k\rangle$ satisfies the same conditions that $\langle g, s\rangle$ satisfies and $k < s$.

This shows a contradiction and establishes that T is well-founded. □

Proof of Claim 2.6.25. Suppose not, then there is a $y_0 \in T$ such that G_{β,y_0} is not trivial. Otherwise, G_β is trivial. Similarly there is a y_1 such that $(y_0, y_1) \in T$ and $G_{\beta,(y_0,y_1)}$ is not trivial. Continue in this way we get an infinite sequence $f = (y_0, y_1, \cdots)$ such that $G_{\beta,f\restriction k}$ is not trivial and $(f \restriction k) \in T$ for each $k \geq 0$. This contradicts the well-foundedness of T. $\square$

This finishes the proof of the main theorem.

2.6.4 Proof of Theorem 2.6.33

Definition 2.6.26. Let $p, q \in Y^{<\omega}$ be two positions. Say that p extends q if there exist a natural number k strictly less than the length of q such that $q = p \restriction k$.

Definition 2.6.27. Two positions $p, q \in Y^{<\omega}$ are compatible if p extends q or q extends p or $p = q$. Otherwise p and q are incompatible.

Let $T \subset Y^\omega$ be a set of positions. Say that p is incompatible with T if p is incompatible with any $q \in T$. Say that T is a set of incompatible positions if p is incompatible with $T \setminus \{p\}$ for any $p \in T$.

Lemma 2.6.28. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leqslant \gamma\rangle$ be a reduction chain. Let $T \subset Y^\omega$ be a set of incompatible positions such that for each $p \in T$ there exists a least ordinal α_p such that $G_{\alpha,p}$ is trivial. Then there exists a reduction chain $\langle H_\alpha \mid \alpha \leqslant \delta\rangle$ such that $H_\delta = G_\gamma$ and H_0 is defined by*

1. *$H_{0,p} = G_{\alpha_p,p}$ for each $p \in T$;*

2. *$H_{0,p} = G_p$ for $p \in Y^{<\omega}$ incompatible with T.*

Proof. For each $\alpha \leq \gamma$, let $T_\alpha = \{p \in T \mid \alpha_p > \alpha\}$. So T_α records those subgames G_p of G with $p \in T$ that haven't become trivial before α.

For each $\alpha \leq \gamma$, define a new game $G'_\alpha = \langle A'_{1,\alpha}, A'_{2,\alpha} \rangle$ as follows. For each $p \in T_\alpha$, let $G'_{\alpha,p} = G_{\alpha_p,p}$; for $q \in Y^{<\omega}$ incompatible with T_α, let $G'_{\alpha,p} = G_{\alpha,p}$.

So we have a new sequence $\langle G'_\alpha = \langle A'_{1,\alpha}, A'_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$. Note that this may not be a reduction chain since it may have repeated terms.

We then delete those repeated terms.

Let $H_0 = G'_0$.

Suppose H_α is defined and $H_\alpha = G'_{\alpha'}$ for some $\alpha' \leq \gamma$, let $H_{\alpha+1}$ to be $G'_{(\alpha+1)'}$ where $(\alpha+1)'$ is the least ordinal $> \alpha'$ such that $G'_{(\alpha+1)'} \neq H_\alpha$.

If such β does not exist, we stop and let $\delta = \alpha$.

If θ is a limit ordinal, let $H_\theta = \lim\limits_{\alpha<\theta} H_\alpha$. Let $\theta' = \sup\{\alpha' \mid \alpha < \theta\}$. Then $H_\theta = \lim\limits_{\alpha<\theta} H_\alpha = \lim\limits_{\alpha<\theta} G'_{\alpha'} = \lim\limits_{\alpha'<\theta'} G'_{\alpha'} = \lim\limits_{\beta<\theta'} G'_\beta = G'_{\theta'}$.

This completes the inductive construction.

We show next that $\langle H_\alpha \mid \alpha \leqslant \delta \rangle$ is the required reduction chain such that $H_\delta = G_\gamma$.

By the definitions of H_0 and H_δ, they are the required games. So we only need to show that it is a reduction chain. Clearly the definition in the limit stages satisfies the requirement of a reduction chain so we shall only check the successor step.

$H_{\alpha+1}$ is $G'_{(\alpha+1)'}$ where $(\alpha+1)'$ is the least ordinal $> \alpha'$ such that $G'_{(\alpha+1)'} \neq H_\alpha$. $G'_{(\alpha+1)'}$ is an application of an axiom to $G'_{(\alpha+1)'-1}$, which is either $G'_{\alpha'} = H_\alpha$ or same as $G'_{\alpha'} = H_\alpha$ by definition. So $H_{\alpha+1}$ is obtained by an application of the same axiom to H_α. □

Lemma 2.6.29. *Suppose that $G = \langle A_1, A_2 \rangle$ can be reduced to G_1 by applying an* *axiom, say Axiom I, at the position $\emptyset$. Let H be obtained G by applying Axiom II (possibly same as Axiom I) to a position $p \neq \emptyset$. Then G_1 can also be obtained from H by applying Axiom I to the position $\emptyset$.*

Proof. We show this case by case.

Case 1. Axiom I is case 1 of refined backward induction. This is impossible since it requires that G_p is already trivial.

Case 2. Axiom I is case 2 of refined backward induction. So there is a y such that that $G_y = \langle N_y, \emptyset \rangle$. But G_p is not trivial, so $y \neq p \restriction 1$. So the axiom is still applicable to H since $H_y = G_y$.

Case 3. Axiom I is cooperation axiom.

Since G_p is not trivial $\tilde{A}_{1,p} = \tilde{A}_{1,p} = N_p \cap \tilde{A}_1$.

Subcase 1. $N_p \cap \tilde{A}_1 = N_p \cap \tilde{A}_2 \neq \emptyset$. Then $\tilde{A}_{1,p} = \tilde{A}_{1,p} = N_p \cap \tilde{A}_1 \neq \emptyset$. Cooperation axiom is applicable to the subgame G_p also. If Axiom II is not the cooperation axiom, by Lemma 2.6.28, it gives the same result as the cooperation axiom. So we assume that Axiom II is the cooperation axiom. So $H_p = \langle N_p, N_p \rangle$ and $\tilde{B}_1 = \tilde{B}_2 = \tilde{A}_1 \cup N_p \neq \emptyset$. So Axiom I, the cooperation axiom, is applicable to H.

Subcase 2. $N_p \cap \tilde{A}_1 = N_p \cap \tilde{A}_2 = \emptyset$. That is $N_p \cap (A_1 \setminus K) = N_p \cap (A_2 \setminus K) = (N_p \cap A_1) \setminus (N_p \cap K) = (N_p \cap A_2) \setminus (N_p \cap K) = \emptyset$. So $A_{1,p} \cup A_{2,p} \subset K$. ($K$ is defined in equation 4.1.) We show case by case that this is impossible.

Axiom II cannot be case 2 of refined backward induction since that would require there exists a y such that the player that moves at this position, say player 1, wins the subgame at $p^\frown y$. I.e., $N_{p^\frown y} \subset A_1$ and p is a position of player 1. This contradicts $A_{1,p} \subset K$.

Axiom II cannot be the cooperation axiom. This is obvious since $\tilde{A}_{1,\gamma,p} = \tilde{A}_{2,\gamma,p} = \emptyset$.

Axiom II is case 1 of refined backward induction, say player 1 moves at p. For refined backward induction to be applicable, it must be true that for all y, $G_{p^\frown y}$ is trivial.

For $\tilde{A}_1 \cap N_p = \emptyset$, it must be true that for all y, $A_{1,p^\frown y} = \emptyset$. Otherwise for some y, $A_{1,p^\frown y} = N_{p^\frown y}$, since $G_{p^\frown y}$ is trivial, either $A_{2,p^\frown y} = N_{p^\frown y}$ or $A_{2,p^\frown y} = \emptyset$. So $\tilde{A}_{1,p} = N_{p^\frown y}$ since G_p is not trivial, a contradiction.

Since G_p is not trivial there is a y' such that $A_{2,p^\frown y'} = N_{p^\frown y'}$ and a y'' such that $A_{2,p^\frown y''} = \emptyset$. So applying the axiom to G_p we have $H_p = \langle \emptyset, \emptyset \rangle$. Then $\tilde{B}_1 = \tilde{B}_2 = \tilde{A}_1 = \tilde{A}_2 \neq \emptyset$. So the the cooperation axiom still applies to H.

Axiom II cannot be case 1 of the third axiom. Otherwise player 1 has a winning strategy, say σ is a t-strategy for player 1 in the subgame G_p. Take any f such that $f \restriction k \in \sigma$ for each $k \geq$ the length of p. Since σ is a winning t-strategy $f \in A_1$. For $\tilde{A}_1 \cap N_p = \emptyset$, then there has to be a k such that $f \restriction k$ is a position for player 2 to move and $N_{f \restriction (k+1)} \subset K$.

We claim that there exists a particular f and a k such that $f \restriction k$ is a position for player 2 to move and $N_{(f \restriction k)^\frown y} \subset K$ for each y. I.e., $N_{(f \restriction k)^\frown y} \cap A_2 = \emptyset$. Since $(f \restriction k)$ is a position for player 2 to move and σ is a winning t-strategy, $N_{(f \restriction k)^\frown y} \subset A_1$. So $G_{f \restriction k}$ is trivial, this contradicts the requirement for $N_{(f \restriction k)^\frown y} \subset K$.

We now show the claim. We assume, towards a contradiction, that for each such f and k there is a y such that $N_{(f \restriction k)^\frown y} \not\subseteq K$, i.e., $A_{1,(f \restriction k)^\frown y} \neq N_{(f \restriction k)^\frown y}$.

Start from $p_0 = p$. p_0 is a position for player 1 to move, so there exists y_0 such that $p_0{}^\frown y_0 \in \sigma$. Let $p_1 = p_0{}^\frown y_0$. By assumption there exists a y_1 such that $A_{1,p_1{}^\frown y_1} \neq N_{p_1{}^\frown y_1}$. Let $p_2 = p_1{}^\frown y_1$. Continue in this manner. Let $g = p_0{}^\frown y_0{}^\frown y_1{}^\frown \cdots$. Then for all k such that $g \restriction k$ is a position for player 2 to move and $A_{1,g\restriction(k+1)}$ is not trivial. So $N_{g\restriction(k+1)} \not\subseteq K$, a contradiction.

Axiom II cannot be case 2 of the third axiom. Otherwise player 1 has a winning strategy, in particular A_1 is not empty. Let $f \in A_1$, then $f \in A_2 = Y^\omega$. Then $f \in \tilde{A}_1$ since for f to be in K there has to be a k and an i such that $N_{f\restriction k} \cap A_i = \emptyset$, but $f \in N_{f\restriction k} \cap A_i$, which is a contradiction. Therefore $f \in \tilde{A}_1$ contradicts $\tilde{A}_1 \cap N_p = \emptyset$.

Case 4. Axiom I is case 1 of the third axiom, say player 1 has a winning strategy. So $A_2 = \emptyset$.

Subcase 1. Suppose player 1 has a t-strategy σ such that $p \in \sigma$, then $\sigma_p = \{q \in \sigma \mid q \text{ extends } p \text{ or } q = p\}$ is a winning strategy for player 1 in the subgame G_p. So G_p is subgame case 1 of the third axiom is applicable. By Lemma 2.6.28, Axiom II applied to the subgame G_p gives the same result as that given by case 1 of the third axiom, if Axiom II is not case 1 of the third axiom. So $H_p = \langle N_p, \emptyset \rangle$ and σ is still a winning strategy for H_p.

Subcase 2. Suppose player 1 has a t-strategy σ such that $p \notin \sigma$, then σ is also a t-strategy for H_p.

Case 5. Axiom I is case 2 of the third axiom. Similar argument as in case 4 works. □

Lemma 2.6.30. *Suppose that $\langle G_\alpha \mid \alpha \leq \gamma \rangle$ is a reduction chain and at each successor step $G_{\alpha+1}$ is obtained from G_α by applying an axiom to a position q_α.*

Suppose further that q_α extends a position p for all $\alpha < \gamma$, i.e., for all $\alpha < \gamma$, $G_{\alpha+1}$ are obtained from G_α by applying an axiom to a subgame of $G_{\alpha,p}$.

Suppose that G can also be reduced to H by applying an axiom, say Axiom I, at the position p. Then Axiom I can also be applied to G_γ to get H.

Suppose that G_γ is trivial, then $G_\gamma = H$.

Proof. We show by induction on $\alpha \leq \gamma$ that Axiom I is applicable to G_α. In particular, if G_γ is trivial, then Axiom I is still applicable to G_γ, which implies $G_\gamma = H$.

By assumption Axiom I is applicable to $G_0 = G$.

Suppose that Axiom I is applicable to G_α. Since $G_{\alpha+1}$ is obtained from G_α by applying an axiom to a subgame, say $G_{\alpha,q}$, where q extends p, of $G_{\alpha,p}$. So Lemma 2.6.29 says that Axiom I is still applicable to $G_{\alpha+1}$.

Suppose θ is a limit ordinal. We then proceed case by case on Axiom I.

Case 1. Axiom I cannot be case 1 of refined backward induction since this requires $G_{p^\frown y}$ for each y is already.

Case 2. Axiom I is case 2 of refined backward induction. Since this requires $G_{0,p^\frown y}$ for some y and this still holds for $G_{\theta,p^\frown y}$, Axiom I is applicable to G_θ.

Case 3. Axiom I is the cooperation axiom.

Reexamining case 3 of the proof of Lemma 2.6.29 we can see that, for each q_α, there are two possibilities: if $\tilde{A}_{1,\alpha} \cap N_{q_\alpha} \neq \emptyset$, then $A_{i,\alpha+1} = A_{i,\alpha} \cup N_{q_\alpha}$ for each i; otherwise only case 1 of refined backward induction applies and $A_{i,\alpha+1} = A_{i,\alpha} \setminus N_{q_\alpha}$ for each i.

Now compare A_i with $A_{i,\theta}$. First let

$$T = \{q_\alpha \mid \forall\beta < \theta, \beta \neq \alpha(q_\beta \text{ extends } q_\alpha \text{ or } q_\beta \text{ is incompatible with } q_\alpha)\}.$$

Define

$$R = \{N_{q_\alpha} \mid q_\alpha \in T \ \& \ \tilde{A}_{1,\alpha} \cap N_{q_\alpha} \neq \emptyset\},$$

and

$$S = \{N_{q_\alpha} \mid q_\alpha \in T \ \& \ \tilde{A}_{1,\alpha} \cap N_{q_\alpha} = \emptyset\}.$$

Then $A_{i,\theta} = (A_i \cup R) \setminus S$.

Let K_α be the K defined in Equation () for the game G_α. Now compare K_0 with K_θ. By definition, we have

$$K_0 = \{N_{q^\frown y} \mid \exists i(i \text{ moves at } q) \ \& \ G_{0,q^\frown y} \text{ is trivial}$$
$$\& \ G_{0,q} \text{ is not trivial} \ \& \ A_{i,0,q} = \emptyset\}.$$

$$K_\theta = \{N_{q^\frown y} \mid \exists i(i \text{ moves at } q) \ \& \ G_{\theta,q^\frown y} \text{ is trivial}$$
$$\& \ G_{\theta,q} \text{ is not trivial} \ \& \ A_{i,\theta,q} = \emptyset\}.$$

$K_\theta = (K_\theta \setminus K_0) \cup (K_0 \cap K_\theta)$.

First consider each $N_{q^\frown y} \subset (K_\theta \setminus K_0)$. There are two reasons that $N_{q^\frown y}$ is in K_θ but not in K_0: at a stage, say α, $G_{\alpha,q^\frown y}$ become trivial, or $A_{i,\alpha,q^\frown y} = \emptyset$, and $G_{\theta,q^\frown y} = G_{\alpha,q^\frown y}$. Note this can happen only when there is a subgame of $G_{\alpha,q^\frown y}$ become trivial at a successor stage $(\beta+1) \leq \alpha$. The condition that q_β, q_β extending $q^\frown y$, satisfies must be the case that $\tilde{A}_{1,\beta} \cap N_{q_\beta} \neq \emptyset$, hence case 1 of refined backward induction applies and $A_{i,\beta+1} = A_{i,\beta} \setminus N_{q_\beta}$ for each i. So $G_{\beta+1,q_\beta} = \langle\emptyset, \emptyset\rangle$. $G_{\beta+1,q_\beta}$

being a subgame of $G_{q^\frown y}$ and $G_{\alpha,q^\frown y}$ is trivial implies that $G_{\theta,q^\frown y} = G_{\alpha,q^\frown y} = \langle\emptyset,\emptyset\rangle$. So $(K_\theta \setminus K) \cap A_{i,\theta} = \emptyset$ for each i. Therefore

$$(K_\theta \setminus K)^c \cap A_{i,\theta} = A_{i,\theta}. \tag{2.7}$$

Next we consider each $N_{q^\frown y} \subset (K_0 \setminus K_\theta)$. Note that once $G_{0,q^\frown y}$ become trivial it has to be trivial forever. So there are two reasons that $N_{q^\frown y}$ is in K_0 but not in K_θ: at a stage, say α, $G_{\alpha,q}$ become trivial, or $A_{i,\alpha,q^\frown y} = N_{q^\frown y}$, and $G_{\theta,q^\frown y} = G_{\alpha,q^\frown y}$. For this to happen, by case 3 of the proof of Lemma 2.6.29, there are two possibilities:

Subcase 1. $G_{\alpha,q}$ become trivial through applications of case 1 of refined backward induction. So $G_{\alpha,q} = \langle\emptyset,\emptyset\rangle$. Therefore $N_{q^\frown y} \subset S$.

Subcase 2. $\alpha = \beta+1$ is a successor ordinal and $G_{\alpha,q}$ become trivial or $A_{i,\alpha,q^\frown y} = N_{q^\frown y}$ because of an application of the cooperation axiom to q_β, $q^\frown y$ extending q_β. So $G_{\alpha,q_\beta} = \langle N_{p_\beta}, N_{p_\beta}\rangle$ and $G_{\alpha,q} = \langle N_p, N_p\rangle$. Therefore $N_{q^\frown y} \subset R$.

So we have $(K_0 \setminus K_\theta) \subset (R \cup S)$. It follows that $(K_0 \setminus K_\theta) = (R \cup S) \cap (K_0 \setminus K_\theta)$ and $S^c \cap (K_0 \setminus K_\theta) = S^c \cap (R \cup S) \cap (K_0 \setminus K_\theta) = R \cap S^c \cap (K_0 \setminus K_\theta)$. Therefore

$$A_i \cap S^c \cap (K_0 \setminus K_\theta) = A_i \cap R \cap S^c \cap (K_0 \setminus K_\theta) \subset R \cap S^c \cap (K_0 \setminus K_\theta). \tag{2.8}$$

Write

$$\begin{aligned} K_\theta &= (K_\theta \setminus K_0) \cup (K_0 \cap K_\theta) \\ &= (K_\theta \setminus K_0) \cup [K_0 \setminus (K_0 \setminus K_\theta)] \\ &= (K_\theta \setminus K_0) \cup [K_0 \cap (K_0 \setminus K_\theta)^c] \end{aligned}$$

Now

$$
\begin{aligned}
\tilde{A}_{i,\theta} &= A_{i,\theta} \setminus K_\theta \\
&= A_{i,\theta} \setminus \big[(K_\theta \setminus K_0) \cup [K_0 \cap (K_0 \setminus K_\theta)^c]\big] \\
&= A_{i,\theta} \cap \big[(K_\theta \setminus K_0) \cup [K_0 \cap (K_0 \setminus K_\theta)^c]\big]^c \\
&= A_{i,\theta} \cap \big[(K_\theta \setminus K_0)^c \cap [K_0^c \cup (K_0 \setminus K_\theta)]\big] \\
&= [A_{i,\theta} \cap (K_\theta \setminus K_0)^c] \cap [K_0^c \cup (K_0 \setminus K_\theta)] \\
&= A_{i,\theta} \cap [K_0^c \cup (K_0 \setminus K_\theta)] \ (\because\ \text{Equation (2.7)}) \\
&= [(A_i \cup R) \setminus S^c] \cap [K_0^c \cup (K_0 \setminus K_\theta)] \\
&= [(A_i \cup R) \cap S^c \cap K_0^c] \cup [(A_i \cup R) \cap S^c \cap (K_0 \setminus K_\theta)] \\
&= [(A_i \setminus K_0) \cup R \cap S^c] \\
&\quad \cup \big[[A_i \cap S^c \cap (K_0 \setminus K_\theta)] \cup [R \cap S^c \cap (K_0 \setminus K_\theta)]\big] \\
&= [(\tilde{A}_i \cup R) \cap S^c] \cup [R \cap S^c \cap (K_0 \setminus K_\theta)] \\
&\qquad (\because\ \text{Equation (2.8)}) \\
&= (\tilde{A}_i \cup R) \cap S^c \\
&= \tilde{A}_i \cup R \ (\because A_{i,\theta} \cap S = \emptyset).
\end{aligned}
$$

So $\tilde{A}_{1,\theta} = \tilde{A}_{2,\theta} \neq \emptyset$ and the axiom still applies.

Case 4. Axiom I is case 1 or case 2 of the third axiom. Say player 1 wins $G_{0,p}$. Suppose, towards a contradiction, that player 1 does not have a winning strategy in $G_{\theta,p}$. Then by Lemma 2.6.21, player 1 loses $G_{\theta,p}$. This contradicts Lemma 2.6.28 since on the one hand there is a chain such that player 1 loses the game $G_{0,p}$ and on the other hand the third axiom can be applied directly to $G_{0,p}$ such that player 1 wins.

This completes the inductive construction. □

Lemma 2.6.31. *Suppose that $\langle G_\alpha \mid \alpha \leq \gamma+1\rangle$ is a reduction chain and at each successor step $G_{\alpha+1}$ is obtained from G_α by applying an axiom to a position q_α. Suppose further that q_α extends a position p for all $\alpha < \gamma$ and $q_\gamma = \emptyset$. I.e., for all $\alpha < \gamma$, $G_{\alpha+1}$ are obtained from G_α by applying an axiom to a subgame of $G_{\alpha,p}$ and $G_{\gamma+1}$ is obtained applying an axiom, say Axiom I, to G_γ.*

Suppose that G can also be reduced to H by applying an axiom, say Axiom II, at a position p. Then Axiom I can also be applied to H to get $G_{\gamma+1}$.

Proof. By Lemma 2.6.30 Axiom II is applicable to G_γ. Then by Lemma 2.6.29, Axiom I is applicable to H. □

Lemma 2.6.32. *Let $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \leqslant \gamma+1\rangle$ be a reduction chain for $G = G_0$ such that in the last step $G_{\gamma+1}$ is obtained by applying an axiom, say Axiom I, to G_γ at the position $\emptyset$. Let $p \neq \emptyset$ be a position such that an axiom, say Axiom II, is applicable to the subgame G_p at the position p. Suppose that no axiom is applied to a subgame $G_{\alpha,q}$ for $q \neq \emptyset$ and q is extended by p. Then there exists a reduction chain $\langle H_\alpha \mid \alpha \leqslant \delta\rangle$ such that $H_\delta = G_\gamma$ and H_0 is defined by*

1. *$H_{0,p} =$ The trivial game obtained by applying the axiom to G_p;*

2. *$H_{0,q} = G_q$ for $q \in Y^{<\omega}$ incompatible with p.*

Proof. By Lemma 2.6.30, Axiom II is applicable to each G_α, $\alpha \leq \gamma$. Let G'_α be the game obtained by applying Axiom II to G_α. By Lemma 2.6.29, Axiom I is applicable to G'_γ. So the chain $\langle G'_\alpha \mid \alpha \leqslant \gamma+1\rangle$ is still a reduction chain except that it may have repeated terms. Delete those repeated terms as in () to get the sequence $\langle H_\alpha \mid \alpha \leqslant \delta \rangle$. $\square$

Theorem 2.6.33. *Let $\mathcal{G} = \langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha} \rangle \mid \alpha \leqslant \gamma \rangle$ and $\mathcal{H} = \langle H_\alpha = \langle B_{1,\alpha}, B_{2,\alpha} \rangle \mid \alpha \leqslant \delta \rangle$ be two reduction chains for the game $G = \langle A_1, A_2 \rangle$ such that $G_0 = H_0 = G$ and G_γ, H_δ are trivial. Then $G_\gamma = H_\delta$.*

Proof. We define by induction, for each $\beta \leq \gamma$, a reduction chain $\mathcal{H}_\beta = \langle H_{\alpha,\beta} = \langle B_{1,\alpha,\beta}, B_{2,\alpha,\beta} \rangle \mid \alpha \leqslant \delta_\beta \rangle$ such that $H_{0,\beta} = G_\beta$ and $H_{\delta_\beta,\beta} = H_\delta$.

In particular, in the sequence $\mathcal{H}_\gamma$, we will have $H_{0,\gamma} = G_\gamma$ and $H_{\delta_\gamma,\gamma} = H_\delta$. But $H_{0,\gamma} = G_\gamma$ is trivial by assumption, so $\delta_\gamma = 0$ and $H_{0,\gamma} = H_{\delta_\gamma,\gamma}$. Therefore $G_\gamma = H_\delta$.

Let $\mathcal{H}_0 = \mathcal{H}$.

Suppose that $\mathcal{H}_\beta = \langle H_{\alpha,\beta} = \langle B_{1,\alpha,\beta}, B_{2,\alpha,\beta} \rangle \mid \alpha \leqslant \delta_\beta \rangle$ has been defined such that $H_{0,\beta} = G_\beta$ and $H_{\delta_\beta,\beta} = H_\delta$. We proceed to define $\mathcal{H}_{\beta+1} = \langle H_{\alpha,\beta+1} = \langle B_{1,\alpha,\beta+1}, B_{2,\alpha,\beta+1} \rangle \mid \alpha \leqslant \delta_{\beta+1} \rangle$ such that $H_{0,\beta+1} = G_{\beta+1}$ and $H_{\delta_{\beta+1},\beta+1} = H_\delta$.

Since $G_{\beta+1}$ is obtained from G_β by applying an axiom to a subgame $G_{\beta,p_\beta} = H_{0,\beta,p_\beta}$, for some $p_\beta \in Y^{<\omega}$.

Let α_{p_β} be the least ordinal such that the subgame $H_{\alpha_{p_\beta},\beta,p_\beta}$ is trivial. Consider two possibilities.

Case 1. α_{p_β} is a limit ordinal. $G_{\beta+1,p_\beta}$ is obtained from G_{β,p_β} by applying an axiom to it. $H_{\alpha_{p_\beta},\beta,p_\beta}$ becomes trivial by a reduction chain for $H_{0,p_\beta} = G_{\beta+1,p_\beta}$. By Lemma 2.6.30. We know that $G_{\beta+1,p_\beta} = H_{\alpha_{p_\beta},\beta,p_\beta}$. Apply Lemma 2.6.28 to the sequence $\mathcal{H}_\beta$ and $T = T_\beta = \{p_\beta\}$ to get a new sequence $\mathcal{H}_{\beta+1}$. It follows from the lemma and the fact $G_{\beta+1,p_\beta} = H_{\alpha_{p_\beta},\beta,p_\beta}$ that $H_{0,\beta+1} = G_{\beta+1}$ and $H_{\delta_{\beta+1},\beta+1} = H_\delta$.

Case 2. $\alpha_{p_\beta} = \alpha + 1$ is a successor ordinal. Let $H_{\alpha,\beta,q}$ be the subgame such that an axiom is applied to it to obtain $H_{\alpha_{p_\beta},\beta}$ from $H_{\alpha,\beta}$. Note that p_β, q cannot be incompatible. Otherwise H_{α,β,p_β} would be trivial since what is happening in the game H_{α,β,p_β} does not affect H_{α,β,p_β} since p_β and q are incompatible. This contradicts the minimality of α_{p_β}. So there are two possibilities.

Subcase 1. q extends p_β or $q = p_\beta$. As in case 1, by Lemma 2.6.30 we know that $G_{\beta+1,p_\beta} = H_{\alpha_{p_\beta},\beta,p_\beta}$ since one is obtained from a direct application of an axiom and the other is obtained from a reduction chain for the same game. Apply Lemma 2.6.28 to the sequence $\mathcal{H}_\beta$ and $T = T_\beta = \{p_\beta\}$ to get a new sequence $\mathcal{H}_{\beta+1}$. It follows from the lemma and the fact $G_{\beta+1,p_\beta} = H_{\alpha_{p_\beta},\beta,p_\beta}$ that $H_{0,\beta+1} = G_{\beta+1}$ and $H_{\delta_{\beta+1},\beta+1} = H_\delta$.

Subcase 2. p_β extends q.

Apply Lemma 2.6.32 to the sequence $\mathcal{H}_\beta$ and the subgame $H_{0,\beta,p_\beta} = G_{\beta,p_\beta}$ to get a new sequence $\mathcal{H}_{\beta+1}$. It follows from the lemma that $H_{0,\beta+1} = G_{\beta+1}$ and $H_{\delta_{\beta+1},\beta+1} = H_\delta$, and also the fact $G_{\beta+1,p_\beta} = H_{\alpha_{p_\beta},\beta,p_\beta}$.

Let θ be a limit ordinal. Let

$$T_\theta = \{p_\beta \mid \beta < \theta \text{ and for all } \beta' < \theta, \text{ either } p_{\beta'} \text{ is}$$
$$\text{incompatible with } p_\beta \text{ or } p_{\beta'} \text{ extends } p_\beta\}.$$

Apply Lemma 2.6.28 to the sequence $\mathcal{H}$ and $T = T_\theta$ to get a new sequence $\mathcal{H}_\theta$.

By the lemma $H_{0,\theta}$ is obtained by replacing the subgames H_{0,p_β} with $H_{\alpha_{p_\beta},\beta,p_\beta}$ for each $p_\beta \in T_\theta$. Note that G_θ is can also be regarded as obtained from G_0 by by replacing the subgames G_{0,p_β} with $G_{\beta+1,p_\beta}$ for each $p_\beta \in T_\theta$. But we have shown in the successor steps that for each $p_\beta \in T_\theta$, $H_{\alpha_{p_\beta},\beta,p_\beta} = G_{\beta+1,p_\beta}$. Therefore, by the lemma, $H_{0,\theta} = G_\theta$ and $H_{\delta_\theta,\theta} = H_\delta$.

This completes the inductive construction. $\square$

2.6.5 Proof of Theorem 2.5.2

Proof. In the following we shall use the set-representation of a strategy σ. σ will consists of elements of the form $\langle p, y\rangle$ where p is a position for player 1 to move and $y \in Y$. σ will satisfy the property that for each p a position for player 1 to move there exists one and only one $y \in Y$ such that $\langle p, y\rangle \in \sigma$. So σ can be transformed into a strategy $S_1 : M_1 \to Y$ by letting for each p a position for player 1 to move, $S_1(p) = y$ where $\langle p, y\rangle \in \sigma$. Since such y exists and there is only one such y, S_1 is well-defined. τ also satisfies similar property and can be transformed into a strategy S_2 for player 2.

We proceed in two steps to build σ and τ.

Step 1.

For any determined game G with a reduction chain $\langle G_\alpha = \langle A_{1,\alpha}, A_{2,\alpha}\rangle \mid \alpha \le \gamma\rangle$ we define a set of partial strategies $\langle \sigma(G), \tau(G)\rangle$ as follows.

For each $p \in Y^{<\omega}$, let α_p be the least ordinal such that $G_{\alpha_p,p}$ is trivial and, $\alpha_p = 0$ or α_p is a limit ordinal or $\alpha_p = \alpha + 1$ is a successor ordinal and G_{α_p} is obtained from G_α by an application of an axiom to the subgame $G_{\alpha,p}$.

We define a pair $\langle \sigma_\alpha, \tau_\alpha\rangle$ for each $\alpha \le \gamma$ inductively such that

1. for all $\beta < \alpha$, $\langle p, y\rangle \in \sigma_\beta$ implies $\langle p, y\rangle \in \sigma_\alpha$.

2. $\langle p, y\rangle \in \sigma_\beta$ and $\langle p, y'\rangle \in \sigma_\alpha$, $\beta \le \alpha$, implies $y' = y$.

3. If $\alpha_p = \alpha$ then there exists an $f_p \in Y^\omega$ such that

   (a) $f_p \restriction k_0 = p$ for some k_0;

   (b) for each $k \geq k_0$, an even number, $\langle f_p \restriction k, f_p(k) \rangle \in \sigma_\alpha$,

   (c) for each $k \geq k_0$, an odd number, $\langle f_p \restriction k, f_p(k) \rangle \in \tau_\alpha$;

   (d) for each i, $f_p \in A_i$ if and only if $A_{i,\alpha_p,p} = N_p$;

   (e) if $f_p \notin A_2$, then for all $f \in Y^\omega$ such that for each $k \geq k_0$, an even number, $\langle f \restriction k, f(k) \rangle \in \sigma_\alpha$, $f \notin A_2$;

   (f) if $f_p \notin A_1$, then for all $f \in Y^\omega$ such that for each $k \geq k_0$, an odd number, $\langle f \restriction k, f(k) \rangle \in \tau_\alpha$, $f \notin A_1$.

We then let $\sigma(G) = \cup_{\alpha \leq \gamma} \sigma_\alpha$ and $\tau(G) = \cup_{\alpha \leq \gamma} \tau_\alpha$.

For each p such that G_p is trivial, i.e., $\alpha_p = 0$, fix an arbitrary set of strategies σ_p and τ_p for player 1 and 2, respectively, in the subgame G_p. Let $\sigma_0 = \{\sigma_p \mid \alpha_p = 0\}$ and $\tau_0 = \{\tau_p \mid \alpha_p = 0\}$. Clearly $\langle \sigma_0, \tau_0 \rangle$ has the required properties.

Suppose that $\langle \sigma_\beta, \tau_\beta \rangle$ has been defined for each $\beta \leq \alpha$ such that the conditions 1-3 are satisfied. We now proceed to define $\langle \sigma_{\alpha+1}, \tau_{\alpha+1} \rangle$ satisfying the requirement.

Consider the axiom, denoted by Axiom I, used in the successor step to obtain $G_{\alpha+1}$ from G_α.

Case 1. Axiom I is refined backward induction. Suppose the axiom is used for a position p for player 1 to move. Let $\sigma_{\alpha+1} = \sigma_\alpha \cup \{\langle p, y \rangle\}$, where y is any element of Y such that $N_{p^\frown y} \subset A_{1,\alpha}$ if such y exists, otherwise take an arbitrary y. Let $\tau_{\alpha+1} = \tau_\alpha$.

Since $G_{\alpha,p^\frown y}$ is trivial, $\alpha_{p^\frown y} < \alpha_p = \alpha$. So $f_{p^\frown y}$ is defined and, by induction hypothesis, $f_{p^\frown y} \in A_i$ if and only if $A_{i,\alpha,p^\frown y} = A_{i,\alpha_{p^\frown y},p^\frown y} = N_{p^\frown y}$. But, by the axiom, $A_{i,\alpha_p,p} = A_{i,\alpha,p} = N_p$ if and only if $A_{i,\alpha,p^\frown y} = N_{p^\frown y}$. Then, clearly, $f_p = f_{p^\frown y} \in A_i$ if and only if $A_{i,\alpha_p,p} = N_p$.

If $f_p \notin A_1$, then, for all y, $A_{1,\alpha,p^\frown y} = A_{1,\alpha_{p^\frown y},p^\frown y} = \emptyset$. By induction hypothesis, $f_{p^\frown y} \notin A_1$ for all y and for all $f \in Y^\omega$ such that such that for each $k \geq (k_0+1)$, an odd number, $\langle f \restriction k, f(k)\rangle \in \tau_\alpha$, $f \notin A_1$. Note that $\tau_{\alpha+1} = \tau_\alpha$, therefore for all $f \in Y^\omega$ such that such that for each $k \geq k_0$, an odd number, $\langle f \restriction k, f(k)\rangle \in \tau_{\alpha+1}$, $f \notin A_1$.

The rest of the requirement can be easily checked.

Case 2. The cooperation axiom is used in a subgame G_p. Pick an $f \in \tilde{A}_{1,\alpha,p} \cap \tilde{A}_{2,\alpha,p}$. Let $p = f \restriction k_0$ for some k_0. Let

$$\sigma_{\alpha+1} = \sigma_\alpha \cup \{\langle f \restriction k, f(k)\rangle \mid k \geq k_0$$
$$\text{is an even integer and, for all } y, \langle f \restriction k, f(k)\rangle \notin \sigma_\alpha\}$$

and

$$\tau_{\alpha+1} = \tau_\alpha \cup \{\langle f \restriction k, f(k)\rangle \mid k \geq k_0$$
$$\text{is an odd integer and, for all } y, \langle f \restriction k, f(k)\rangle \notin \tau_\alpha\}.$$

Subcase 1. There exists an integer $k_1 \geq k_0$, chosen to be the largest possible, such that for all y, $\langle f \restriction k_1, y\rangle \notin \sigma_\alpha \cup \tau_\alpha$. Let $q = f \restriction k_1$, then $G_{\alpha,q}$ is trivial and f_q is defined such that $f_q \in A_i$ if and only if $A_{i,\alpha,q} = A_{i,\alpha_q,q} = N_q$. But $f \in \tilde{A}_{1,\alpha,p} \cap \tilde{A}_{2,\alpha,p}$, so $f \in A_{i,\alpha,q}$ and $A_{i,\alpha,q} \neq \emptyset$. Therefore $A_{i,\alpha_q,q} = N_q$ and $f_q \in A_i$ for each i. Let $f_p = f_q$. Then for each i, $A_{i,\alpha+1,p} = A_{i,\alpha_p,p} = N_p$ (by the axiom) and $f_p = f_q \in A_i$.

Subcase 2. For all $k \geq k_0$ and for all y, $\langle f \restriction k, y\rangle \notin \sigma_\alpha \cup \tau_\alpha$. Let $f_p = f$. Then clearly $f_p \in A_i$ and $A_{i,\alpha_p,p} = A_{i,\alpha+1,p} = N_p$ for each i.

For the cooperation axiom, conditions 3(e) and 3(f) are vacuously satisfied.

Case 3. Axiom I is case 1 of the third axiom. Say player 1 has a winning strategy in a subgame G_p. Let σ' be a wining strategy for player 1 in G_p and τ' be an arbitrary strategy for player 2 in G_p. Let

$$\sigma_{\alpha+1} = \sigma_\alpha \cup \{\langle q, y\rangle \mid \langle q, y\rangle \in \sigma' \text{ and,}$$
$$\text{for all } y \text{ and for all } q' \text{ extended by } q, \langle q', y\rangle \notin \sigma_\alpha\}$$

and

$$\tau_{\alpha+1} = \tau_\alpha \cup \{\langle q, y\rangle \mid \langle q, y\rangle \in \sigma' \text{ and,}$$
$$\text{for all } y \text{ and for all } q' \text{ extended by } q, \langle q', y\rangle \notin \tau_\alpha\}.$$

Let $f \in Y^\omega$ such that $\langle f \restriction k, f(k)\rangle \in \sigma' \cup \tau'$ for each $k \geq k_0$, where k_0 is the length of p.

Subcase 1. There exists an integer $k_1 \geq k_0$, chosen to be the largest possible, such that for all y, $\langle f \restriction k_1, y\rangle \notin \sigma_\alpha \cup \tau_\alpha$. Let $q = f \restriction k_1$, then $G_{\alpha,q}$ is trivial and f_q is defined such that $f_q \in A_i$ if and only if $A_{i,\alpha,q} = A_{i,\alpha_q,q} = N_q$. But $f \in A_{1,\alpha,p}$ since σ' is a winning strategy. So $f \in A_{1,\alpha,q}$ and $A_{1,\alpha,q} \neq \emptyset$. Therefore $A_{1,\alpha_q,q} = N_q$ and $f_q \in A_1$. Let $f_p = f_q$. Then $A_{1,\alpha+1,p} = A_{1,\alpha_p,p} = N_p$ (by the axiom) and $f_p = f_q \in A_1$. That $f_p \notin A_2$ and $A_{2,\alpha+1,p} = \emptyset$ is obvious since $A_2 = \emptyset$.

Subcase 2. For all $k \geq k_0$ and for all y, $\langle f \restriction k, y\rangle \notin \sigma_\alpha \cup \tau_\alpha$. Let $f_p = f$. Then $f_p \in A_1$ since σ' is a winning strategy and $A_{1,\alpha_p,p} = A_{1,\alpha+1,p} = N_p$ by the axiom. That $f_p \notin A_2$ and $A_{2,\alpha+1,p} = \emptyset$ is obvious since $A_2 = \emptyset$.

For this case, conditions 3(e) and 3(f) are trivially satisfied.

Case 4. Axiom I is case 2 of the third axiom, similar to case 3.

Let θ be a limit ordinal. Fix a $y_0 \in Y$. Let

$$\sigma_\theta = (\cup_{\alpha<\theta}\sigma_\alpha) \cup \{\langle p, y_0\rangle \mid p \in M_1, \alpha_p = \theta \text{ and } \forall\alpha < \theta\forall y \in Y, \langle p, y\rangle \notin \sigma_\alpha\}$$

and

$$\tau_\theta = (\cup_{\alpha<\theta}\tau_\alpha) \cup \{\langle p, y_0\rangle \mid p \in M_2, \alpha_p = \theta \text{ and } \forall\alpha < \theta\forall y \in Y, \langle p, y\rangle \notin \tau_\alpha\}.$$

For each p such that $a_p = \theta$, construct $f_p \in Y^\omega$ inductively as follows.

Let $p_0 = p$.

If p_n is constructed and $\alpha_{p_n} = \theta$. Let $p_{n+1} = p_n^\frown y_0$. Otherwise, $\alpha_{p_n} < \theta$, let $f_p = f_{p_n}$. Since no axiom is applied to subgames of the form G_q, where $q = p_k$ for some $k \leq n$, $G_{\theta,p_n} = G_{\alpha_{p_n},p_n}$. Since $G_{\theta,p}$ is trivial, so $A_{i,\theta,p} = N_p$ if and only if $A_{i,\theta,p_n} = N_{p_n}$. By induction hypothesis, $f_{p_n} \in A_i$ if and only if $A_{i,\alpha_{p_n},p_n} = N_{p_n}$. Therefore $f_p \in A_i$ if and only if $A_{i,\alpha_p,p} = A_{i,\theta,p} = N_p$.

Suppose that for all $n < \omega$, $\alpha_{p_n} = \theta$. Let $f_p = p^\frown y_0^\frown y_0^\frown \cdots$. Since no axiom is applied to subgames of the form G_{p_n} for some n, $f_p \in A_{i,\theta,p}$ if and only if $f \in A_i$. Since $G_{\theta,p}$ is trivial, so $A_{i,\theta,p} = N_p$ if and only if $f_p \in A_{i,\theta,p}$. Therefore $f_q \in A_i$ if and only if $A_{i,\alpha_p,p} = A_{i,\theta,p} = N_p$.

Suppose that $f_p \notin A_i$ for, say, $i = 1$. Then $A_{1,\alpha_p,p} = A_{1,\theta,p} = \emptyset$. Fix an $f \in Y^\omega$ such that for each $k \geq k_0$, an odd number, $\langle f \restriction k, f(k)\rangle \in \tau_\theta$.

Subcase 1. If there exists a $k_1 \geq k_0$ such that $\langle f \restriction k_1, f(k_1)\rangle \in \sigma_\theta \cup \tau_\theta$ and $\alpha_{f\restriction k_1} < \theta$. Choose k_1 to be the least possible. Since no axiom has been applied to subgames of the form $G_{\alpha,f\restriction k}$ for some $k_0 \leq k < k_1$ and for all $\alpha_{f\restriction k_1} < \alpha < \theta$, $A_{1,\alpha_{f\restriction k_1},f\restriction k_1} = A_{1,\theta,f\restriction k_1} = \emptyset$. By induction hypothesis for condition 3(d), $f_{f\restriction k_1} \notin A_1$. By induction hypothesis for condition 3(e,f), for all $g \in Y^\omega$ such that such that for each $k \geq (k_1 + 1)$, an odd number, $\langle g \upharpoonright k, g(k) \rangle \in \tau_{\alpha_{f \upharpoonright k_1}}$, $g \notin A_1$. In particular, f satisfies the condition of g since $\tau_{\alpha_{f \upharpoonright k_1}} \subset \tau_\theta$. So $f \notin A_1$.

Subcase 2. If for all $k \geq k_0$, $\alpha_{f \upharpoonright k} = \theta$. Since no axiom has been applied to subgames of the form $G_{\alpha, f \upharpoonright k}$ for all $k > k_0$ and for all $\alpha < \theta$, $f \in A_{1,\theta,p}$ if and only if $f \in A_1$. By assumption $A_{1,\alpha_p,p} = A_{1,\theta,p} = \emptyset$, so $f \notin A_{1,\theta,p}$. Therefore $f \notin A_1$.

So conditions 3(e,f) still holds at θ.

Step 2.

Build strategies σ and τ step by step. Define the sequences $\langle \sigma_\alpha \mid \alpha \leq \gamma \rangle$, $\langle \tau_\alpha \mid \alpha \leq \gamma \rangle$, γ to be determined, by induction. Then we let $\sigma = \cup \{\sigma_\alpha \mid \alpha \leq \gamma\}$, $\tau = \cup \{\tau_\alpha \mid \alpha \leq \gamma\}$.

Let $\sigma_0 = \tau_0 = \emptyset$.

Suppose $\sigma_\alpha, \tau_\alpha$ have been defined. Suppose that $\sigma_\alpha, \tau_\alpha$ are strategies for player 1 and 2 already, i.e., for each $p \in Y^{<\omega}$ a position for player 1 to move there exists some y such that $\langle p, y \rangle \in \sigma_\alpha$ and for each $p \in Y^{<\omega}$ a position for player 2 to move there exists some y such that $\langle p, y \rangle \in \tau_\alpha$. We then stop and let $\gamma = \alpha$.

Otherwise pick some p with smallest possible length such that p is not in the domain of σ_α or τ_α. That is, say p is a position for player 1 to move, $\langle p, y \rangle \notin \sigma_\alpha$ for all $y \in Y$. G_p is determined by strict determinacy of G. Let $\sigma_{\alpha+1} = \sigma_\alpha \cup \sigma(G_p)$ and $\tau_{\alpha+1} = \tau_\alpha \cup \tau(G_p)$.

If θ is a limit ordinal, let $\sigma_\theta = \cup \{\sigma_\alpha \mid \alpha < \theta\}$, $\tau_\theta = \cup \{\tau_\alpha \mid \alpha < \theta\}$.

To define σ and τ we only need to define an instruction $\langle p, y \rangle$ for σ or τ for each $p \in Y^{<\omega}$. So we need at most $|Y^{<\omega}|^+$ many steps of constructions, here $|Y^{<\omega}|^+$ is the next cardinal after the cardinality of the set $Y^{<\omega}$. So the process terminates at some stage $\gamma \leq |Y^{<\omega}|^+$.

Conclusion.

$\langle \sigma, \tau \rangle$ is a subgame perfect Nash equilibrium with the desired properties. Let $f = \sigma * \tau$, it follows from 3(d) that $f \in A_i$ if and only if $A_{i,\gamma} = Y^\omega$. For each $p \in Y^{<\omega}$, it follows from 3(e) and 3(f) that $\langle \sigma, \tau \rangle$, with the domain restricted to the subgame G_p, forms a Nash equilibrium. □

2.6.6 Proof of Theorem 2.6.6

Proof. For each case we provide an example in which none of rest of the axioms is applicable to a nontrivial subgame of it.

Case 1 of refined backward induction. Let $Y = \{\mathrm{L}, \mathrm{R}\}$, $A_1 = \emptyset$, and $A_2 = N_{\mathrm{L}}$ (see Figure 5).

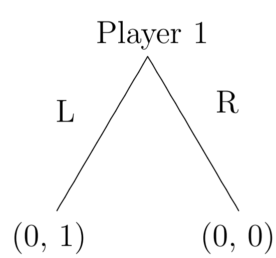

FIGURE 5

Case 2 of refined backward induction. Let $Y = \{\text{Continue}, \text{Stop}\}$,

$$A_1 = \cup\{N_p \mid p = (y_0, y_1, \cdots, y_{2m})$$

$$\& \ y_k = \text{Continue for } k < 2m \ \& \ y_{2m} = \text{Stop}\},$$

and

$$A_2 = \cup\{N_p \mid p = (y_0, y_1, \cdots, y_{2m+1})$$

$$\& \ y_k = \text{Continue for } k < 2m+1 \ \& \ y_{2m+1} = \text{Stop}\}.$$

This game is depicted in Figure 6.

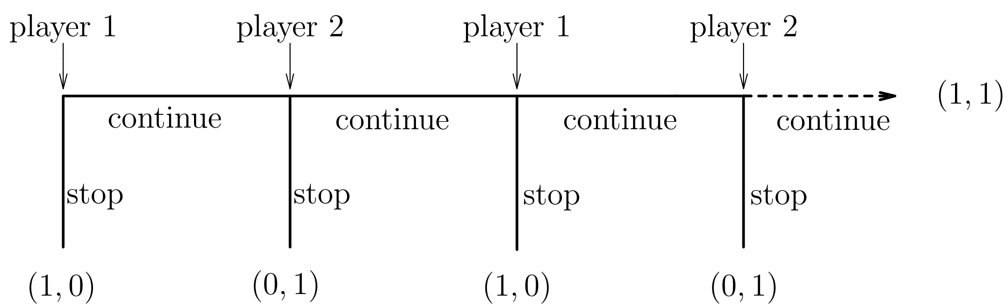

FIGURE 6

The cooperation axiom. Consider the game in Figure 2.

Case 1 of refined backward induction. Let $Y = \{0,1\}$, $A_1 = Y^\omega$. Let σ be a t-strategy of player 2. Define A_2 by letting, for each $f \in Y^\omega$, $f \in A_2$ if and only if, for each $k \geq 0$, $(f \restriction k) \in \sigma$.

Case 2 of refined backward induction, similar to case 1. $\square$

Chapter 3

PI-Games with Infinitely Many Players

3.1 Introduction

Many of the dynamic models in economics involve an infinite number of individuals whose interactions play a crucial role. For example in macroeconomics, quite often the model is infinite horizonal. In the model infinite generations of individuals are involved and decisions of current generations have an impact on that of the future generations. Ideally one would want to formulate this using a pure game-theoretic framework and apply the game theoretic solution concepts to derive the relevant economic outcome. But neither the framework of games with an infinite number of players is founded nor are appropriate solution concepts available.

In this chapter, we shall develop the notion of a perfect information game with an infinite number of players. We shall also define a solution concept for this, namely the notions of determinacy, value and rational strategies. All these are natural extensions of the theory developed for infinite PI-games with a finite number of players.

The rest of this chapter is organized as follows. We present the framework of an infinite PI-game with an infinite number of players in section 2. The solution concept for this PI-game is given in section 3. In section 4 we apply these to an overlapping generation model. Section 5 concludes the chapter.

3.2 PI-games with an Infinite Number of Players

In this section we give the formal representation of a PI-game with an infinite number of players. We shall also define the related notion of a strategy.

Let Y be a set of finite sequences such that if a finite sequence

$$p = (y_0, y_1, \cdots, y_m)$$

is in Y, then

1. for each $k \leqslant m$, the initial segment $(y_0, y_1, \cdots, y_{k-1})$, denoted by $p \restriction k$, is also contained in Y; $p \restriction 0$ is understood to be the empty sequence $\emptyset$;

2. there exists some y_{m+1}, such that the sequence $(y_0, y_1, \cdots, y_m, y_{m+1})$, denoted by $p^\frown y_{m+1}$, is in Y.

Y is intended to be a game tree. An element p of Y is also called a position. For convenience sometimes we also write $y_0 y_1 \cdots y_m$ for a position $(y_0, y_1, \cdots, y_m)$ in Y. A nonempty subset T of Y satisfying condition 1 is called a tree.

Remark 3.2.1. *Since Y is intended to be a game tree, so condition 1 is natural. Condition 2, requiring that each path has to be infinite, is merely for technical convenience. If there is a $(y_0, y_1, \cdots, y_m) \in Y$ without any $(y_0, y_1, \cdots, y_m, y_{m+1}) \in Y$, we can simply add to Y, without changing its game theoretic content, the following elements: $(y_0, y_1, \cdots, y_m, 0)$, $(y_0, y_1, \cdots, y_m, 0, 0)$, etc.*

Let $\mathfrak{F}(Y)$ be the collection of all infinite sequences

$$f = (y_0, y_1, \cdots)$$

such that for all $m \geq 0$, $f \restriction m = (y_0, y_1, \cdots, y_{m-1}) \in Y$.

Let

$$I = \{0, 1, 2, \cdots\}$$

be the collection of all natural numbers. We shall use I to denote the set of players in the game. Hence each player is labeled by a natural number.

Let

$$\lambda : Y \to I$$

be a surjective map. The map λ is a rule that assigns a player $\lambda(p)$ to move at a position $p \in Y$. Hence p is also called a position for player $\lambda(p)$ to move. λ is surjective so that there is no vacuum players, i.e., each player moves at least once.

Each player $i \in I$ is also assigned a payoff function

$$\Phi_i : \mathfrak{F}(Y) \to \mathbb{R}.$$

An infinite PI-game with an infinite number of players G consists of the following data

$$\langle Y, \lambda, (\Phi_i, i \in I) \rangle.$$

A play of the game is run as follows. First player $\lambda(\emptyset)$ begins by choosing y_0, with $(y_0) \in Y$, next player $\lambda(y_0)$ chooses y_1 such that $(y_0, y_1) \in Y$, then player $\lambda(y_0 y_1)$ chooses y_3 with $(y_0, y_1, y_3) \in Y$, etc. Thus an infinite sequence

$$(y_0, y_1, \cdots)$$

is specified. The payoff of player i after this play is $\Phi_i(y_0, y_1, \cdots)$ for each $i \in I$.

Remark 3.2.2. *A player i that ceases to be active after certain moves, and his payoff is constant from p onwards, can be interpreted as a player exits the game.*

A strategy S_i for player i is a subset of Y such that

1. $\emptyset \in S_i$;

2. let $(y_0, y_1, \cdots, y_m) \in S_i$, if it is not a position for player i to move, then

$$(y_0, y_1, \cdots, y_m, y_{m+1}) \in Y \implies (y_0, y_1, \cdots, y_m, y_{m+1}) \in S_i;$$

   otherwise there exists a unique element, denoted by $S_i(y_0, y_1, \cdots, y_m)$, with

$$(y_0, y_1, \cdots, y_m, S_i(y_0, y_1, \cdots, y_m)) \in Y$$

   such that

$$(y_0, y_1, \cdots, y_m, S_i(y_0, y_1, \cdots, y_m)) \in S_i;$$

3. S_i contains no other elements.

Remark 3.2.3. *Condition 1 assures that S_i is not empty and starts the recursive definition in condition 2. If $(y_0, y_1, \cdots, y_m)$ is a position for i to move, S_i is required to specify a move; otherwise all possibilities should be allowed. By abuse of notions,* *we also use S_i as a function defined on all positions that player i is required to move, to all possible moves. Condition 3 says that these are all that a strategy need to specify. In particular, a strategy need not specify moves at inconsistent positions. For example, if player $\lambda(\emptyset)$ decides to play y_0 initially, then he does not need to consider the moves at positions like $(y_0', y_1, \cdots, y_{n-1},)$ where $y_0' \neq y_0$. A formal statement of condition 3 is that S_i is the intersection of all subsets of Y satisfying condition 1 and 2.*

Let $S = \langle S_i \mid i \in I \rangle$ be a set of strategies for the players. A play of the game following the strategies S looks like this:

- First player $\lambda(\emptyset)$ plays, according to $S_{\lambda(\emptyset)}$, a move

$$y_0 = S_{\lambda(\emptyset)}(\emptyset);$$

- then player $\lambda((y_0))$ responds by, according to $S_{\lambda(y_0)}$, a move

$$y_1 = S_{\lambda(y_0)}(y_0);$$

- the game continues so that an infinite sequence

$$(y_0, y_1, \cdots)$$

  is specified.

Denote by $\wedge_{i \in I} S_i$ the resulting play $(y_0, y_1, \cdots)$ according to S.

Let $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$ be a game and let $(y_0, y_1, \cdots, y_m) \in Y$ be a position, we shall define the subgame of G at $p = (y_0, y_1, \cdots, y_m)$. Let

$$Y_p = \{f \mid (y_0, y_1, \cdots, y_m, f(0), f(1), \cdots) \in Y\}.$$

For each $q \in Y_p$, let

$$\lambda_p(q) = \lambda(p^\frown q).$$

Let

$$I_p = \{\lambda_p(q) \mid q \in Y_p\}.$$

For each $i \in I_p$ and for each $f \in \mathfrak{F}(Y)$, let

$$\Phi_{i,p}(f) = \Phi_i((y_0, y_1, \cdots, y_m, f(0), f(1), \cdots)).$$

The subgame of G at $p = (y_0, y_1, \cdots, y_m)$ is the game

$$G_p = \langle Y_p, \lambda_p, (\Phi_{i,p}, i \in I_p)\rangle.$$

Let S_i be a strategy for player i in G and $i \in I_p$, it induces a strategy $S_{i,p}$ on G_p by

$$q \in S_{i,p} \iff p^\frown q \in S_i.$$

The topology on $\mathfrak{F}(Y)$ is defined by taking the basic open neighborhoods to be the form

$$N_{(y_0, y_1, \cdots, y_m)} = \{f \in \mathfrak{F}(Y) \mid f(0) = y_0, \cdots, f(m) = y_m\},$$

for each $(y_0, y_1, \cdots, y_m) \in Y$.

3.3 The Definition

3.3.1 Overview

A game $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$ is called simple if each Φ_i is a characteristic function, i.e., there exists some $X_i \subset \mathfrak{F}(Y)$ such that

$$\Phi_i(f) = \begin{cases} 1 & \text{if } f \in X_i \\ \\ 0 & \text{if } f \notin X_i. \end{cases}$$

In this case, we usually write $G = \langle Y, \lambda, (X_i, i \in I)\rangle$ by replacing the payoff functions by the underlying sets, X_is. And we shall also use the expression that player i receives payoff 1 (or 0) and that player i wins (or loses) interchangely. A simple game $G = \langle Y, \lambda, (X_i, i \in I)\rangle$ is closed if each X_i is a closed set. We assume throughout that all simple games are closed unless otherwise stated.

A game $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$ is said to be continuous if each Φ_i is a continuous function.

By the determinacy result of a game G, or determinacy of G, we mean a collection of answers to the following questions:

1. Is G determined? If yes,

2. What is the value of G, i.e., what is $v(G)$?

3. Can we define rational strategies for player i, $i \in I$? If yes, what are they?

As we shall see later, 1 and 2 are actually equivalent, i.e., a game is determined if and only if the value of the game is defined. If the answer to question 3 is also yes, i.e., there exists at least one set of rational strategies $\langle S_i \mid i \in I\rangle$, then G is called strictly determined.

The goal of this chapter is to define the notion of determinacy for general PI-game $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$.

This goal will be accomplished in two steps. In step 1, we shall define determinacy for simple closed games. In step 2 we define determinacy for continuous games. The connection between step 1 and 2 is that we shall first reduce the determinacy of continuous game $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$ to determinacy of those simple closed games $\langle Y, \lambda, (X_i, i \in I)\rangle$, where the X_is are defined by

$$X_i = \{f \mid f \in \mathfrak{F}(Y) \ \& \ \Phi_i(f) \geq a\}, \tag{3.1}$$

and then use the results of step 1.

3.3.2 Step 1

A quasisolution is an assignment of a subset a_p of I for each $p \in Y$ such that

1. $\lambda(p) \in a_p \iff \exists y(p^\frown y \in Y \wedge \lambda(p) \in a_{p^\frown y})$;

2. $i \neq \lambda(p)$, $i \in a_p \iff \forall y(p^\frown y \in Y \wedge \lambda(p) \in a_{p^\frown y} \implies i \in a_{p^\frown y}) \wedge (\forall y(p^\frown y \in Y \wedge \lambda(p) \notin a_{p^\frown y}) \implies \forall y(i \in a_{p^\frown y}))$.

Now associate to each quasisolution $(a_p)_{p\in Y}$ a subtree T of Y.

1. $\emptyset \in T$

2. if $p \in T$ and $\lambda(p) \notin a_p$, then $p^\frown y \in T$ for all y such that $p^\frown y \in Y$;

3. if $p \in T$ and $\lambda(q) \in a_p$, then $p^\frown y \in T$ for all y such that $p^\frown y \in Y$ and $\lambda(q) \in a_{p^\frown y}$.

Similarly define T_q for each $q \in Y$.

A quasisolution is a solution if, for each $q \in Y$, the following two conditions are satisfied.

1. Consistency requirement.

$$i \in a_q \iff |T_q| \subset X_i.$$

2. There exists no other quasisolution $(b_p)_{p \in Y_q}$ for G_q satisfying the consistency requirement and $b_q \supsetneq a_q$, $b_p \supseteq a_p$ for all $p \in Y_q$.

Definition 3.3.1. A game G is determined if there exists a unique solution.

Definition 3.3.2. If G is determined, the value of player i in G, denoted $v_i(G)$, is 1 if $i \in a_\emptyset$ and it is 0 otherwise.

3.3.3 Step 2

Finally we are in a position to define determinacy for a general PI-game $G = \langle Y, \lambda, (\Phi_i, i \in I)\rangle$. Assume throughout the rest of this chapter that for each $a = (a_i)_{i \in I}$, where each $a_i \in \mathbb{R} \cup \{-\infty, +\infty\}$, the induced simple game $G(a)$ defined with payoff sets

$$X_i = \{f \mid f \in \mathfrak{F}(Y) \ \&\ \Phi_i(f) \geq a_i\},$$

for each i, is determined. We assume in this section that G_a is determined for any a.

We say that $a = (a_i)_{i \in I}$ is secured if $v_i(G(a)) = 1$ for each i.

Let $p = (y_0, y_1, \cdots, y_m) \in Y$. Define in the subgame G_p a set

$$D_p = \big\{ a_p = (a_{p,i})_{i \in I} \mid a_p \text{ is secured } \&$$

$$\forall a'_p = (a'_{p,i})_{i \in I} \big(a'_p \text{ is secured } \implies (\forall i \in I)(a'_{p,i} \leqslant a_{p,i}) \big) \big\}.$$

Define D'_p as the collection of all $(b_i)_{i \in I}$ such that

1. if p is a position for player i to move,

$$b_i = \sup\{ a_{p^\frown y_{m+1},i} \mid a_{p^\frown y_{m+1}} \in D_{p^\frown y_{m+1}} \}.$$

2. Now consider the case $j \neq i$. If there are some $a_{p^\frown y_{m+1}} \in D_{p^\frown y_{m+1}}$ such that

$$b_i = a_{p^\frown y_{m+1},i},$$

   then

$$b_j = \inf\{ a_{p^\frown y_{m+1},j} \mid a_{p,i} = a_{p^\frown y_{m+1},i} \};$$

   Otherwise,

$$b_j = \liminf_{n \to \infty} \{ a_{p^\frown y_{m+1},j} \mid a_{p,i} - a_{p^\frown y_{m+1},i} < 1/n \};$$

Let

$$\bar{D}_p = D_p \cup D'_p.$$

Now define for game G a consistent solution s as follows. Let

$$s \subset \cup_{p \in Y} \bar{D}_p,$$

so that for all $p = (y_0, y_1, \cdots, y_m) \in Y$, s satisfies the following properties

1. there exists a unique $a_p \in s$.

2. if p is a position for player i to move, and if $a_p \in s$, then for all $p^\frown y_{m+1} \in Y$ and $a_{p^\frown y_{m+1}} \in s$,

$$a_{p,i} = \sup\{a_{p^\frown y_{m+1},i} \mid p^\frown y_{m+1} \in Y\}. \tag{3.2}$$

3. Now consider the case $j \neq i$. If there are some $a_{p^\frown y_{m+1}} \in s$ such that

$$a_{p,i} = a_{p^\frown y_{m+1},i},$$

then

$$a_{p,j} = \inf\{a_{p^\frown y_{m+1},j} \mid a_{p,i} = a_{p^\frown y_{m+1},i}\};$$

Otherwise, let

$$a_{p,j} = \liminf_{n\to\infty}\{a_{p^\frown y_{m+1},j} \mid a_{p,i} - a_{p^\frown y_{m+1},i} < 1/n\};$$

Denote by $\mathfrak{s}$ the collection of all such consistent solutions. If $\mathfrak{s}$ is a singleton, write $\mathfrak{s}$ for the s in $\mathfrak{s}$, by abuse of notations.

Definition 3.3.3. The game G is called *determined* if $\mathfrak{s}$ is a singleton. And $a_\emptyset \in \mathfrak{s}$ is called the *value* of the game. If, in addition,

$$\mathfrak{s} \subset \cup_{p\in Y} D_p,$$

then G is called *strictly determined*. In that case, call a strategy S_i for player i *rational* if for all $p = (y_0, y_1, \cdots, y_m)$ a position for player i to move, $p^\frown S_i(p)$ is in one of the rational strategies of player i in the simple game $G_p(a_p)$.

3.4 An Application

Consider the following simple overlapping generation model. $T = 0, 1, 2, \cdots$. At each period t, $t \geq 1$, an individual g_t is born, who lives for two periods, t and $t+1$. At period t g_t is young, and g_t is old at $t+1$. An old individual g_0 exists at period 1.

When an individual is born at period t he is endowed with one unit of good which he can either consume at period t or store for next period's consumption. There is a cost for saving goods for next period's consumption, a fraction of $1-\delta$ will be decayed. So if an individual saves x amount of good he will have δx available for consumption in the next period. Denote by $x_{t,1}$ and $x_{t,2}$ the consumption of individual g_t at period t and period $t+1$. Suppose each g_t has the utility function $u_t(x_{t,1}, x_{t,2}) = \min\{x_{t,1}, x_{t,2}\}$. The old individual at period 0 holds a special indivisible good M which does not decay over time and is equivalent to a fraction, say D, where $0 < D \leqslant 1/(1+\delta)$, of the ordinary good that other individuals have for consumption. And the utility of g_0 depends entirely on his consumption at this period.

One possible pattern of the economy is that each individual consumes his own storage at the period when he is old. So the optimal amount of storage is $1/(1+\delta)$. And each individual gets a utility of $\delta/(1+\delta)$.

Instead of directly consuming their own goods, it is possible for the old and young individuals at period 1 to exchange certain amount of goods to increase their utility. Suppose that they do want to exchange, they have to do so at the fixed price 1/2 of the ordinary good, i.e., individual g_1 gives half of what he have for M.

Once g_1 has M at his hand, he will store nothing but M for next period if he is sure that the same thing is going to happen next, i.e., individual g_2 born at the second period will exchange half of what he have for M so that g_1 can again consume 1/2 of the ordinary good. Thus his total utility will increase from $\delta/(1+\delta)$ to 1/2.

Having described the model and with this background in mind, now we rephrase everything in game theoretic terminology. This would be an infinite-horizon game with an infinite number of players. Again for simplicity, let's assume that $(1/2, 1/2)$ and $(\delta/(1+\delta), 1/(1+\delta))$ are the only possible divisions of the ordinary good.

Period 1, stage 1: g_0 moves first. He has two choices: to consume M or to offer an opportunity to g_1 to exchange. If he chooses to consume M himself, the game ends, all the individuals g_t, $t \geq 1$ has a utility $\delta/(1+\delta)$. If he takes the second option, it is g_1's turn to respond.

Period 1, stage 2: g_1 responds. He has two choices: to exchange or to say "no". If he chooses to exchange, he has to give 1/2 of what he has to g_0 for M. So the utility for g_0 in this game is 1/2. g_1 consumes the other half of what he has and enters period 1 with M. If he takes the second option, the story has the same ending as the case that g_0 choosing to consume M in the stage 1.

Period 2, stage 1: The same story continues with g_1, g_2 taking the roles of g_0, g_1. And the game continues in this way.

We abbreviate the actions by C and E. For the individuals holding M, C means to consume the good M, E means to offer the chance for the young individual just born for exchange. For an individual that was offered such an opportunity, C means to reject the offer, E means to accept the offer. Consider the following two strategies:

Strategy 1 An individual always chooses "E";

Strategy 2 An individual plans to choose "C" at his turn as long as the game continues that long.

One can verify directly that both players playing strategies 1 and both players playing strategies 2 are subgame perfect Nash equilibrium of the game. But, if we employ the notion of determinacy to investigate this game, we can see that this game is determined and the value of the game is that of outcome of the first equilibrium, namely the monetary equilibrium.

Remark 3.4.1. *The special good M in the model captures the following features of money.*

1. *M has certain value, or if an individual wishes he can exchange M with something that has value. This is modeled directly by that M is consumable.*
2. *M is durable or the cost of saving it is lower than that for other goods. This is modeled by that M does not decay.*
3. *M is never consumed directly in the monetary equilibrium.*

Remark 3.4.2. *Dynamic inefficiency. The usual approach to dynamic inefficiency is to introduce a planner that has the power to allocate goods in different generations.*

The outcome of Pareto improvement in the current model is the result of collective intelligence, i.e., all the individual are rational and plays the monetary equilibrium.

Remark 3.4.3. *It is interesting to note that when $D < \delta/(1+\delta)$, to exchange will not be an optimal strategy if only a finite generations live in the world.*

Remark 3.4.4. *This model is basically a modification of Samuelson (1958). We formulate it as a PI-game with an infinite number of players and apply the notion of determinacy to derive the unique outcome.*

Remark 3.4.5. *Another part of the model that differs from Samuelson (1958) is that money is endogenously derived rather than introduced in an ad hoc manner. One disadvantage of using valueless money (goods that is not a source of utility) is that at each period each new generation will have the incentive to issue their own money and old money will not be accepted.*

Chapter 4

Effective Determinacy

4.1 Introduction

Turing machine has been one of the main tools in modeling bounded rationality in game theory. Intuitively, a Turing machine is a computer program that can be implemented by an ideal computer that is different from a usual computer only in that it is assumed to have unbounded memory. The main idea is that in many contexts, the players are not capable of playing strategies of arbitrary complexities. A natural idea of a strategy being simple is a strategy that is implementable by a Turing machine. Hence the class of Turing machine implementable strategies is a natural restriction from bounded rationality perspective.

Besides being a useful tool in modeling bounded rationality, there are potentially practical uses of considering Turing machine implementable strategies. In many practical situations the agents playing the game, like computers, machines, robotics, are not capable of playing arbitrary strategies. And Turing machine implementable strategies would be the ideal class of strategies to consider.

The literature has focused on effectivizing equilibrium concepts on normal form games and repeated games.

In the case of normal form game, the effectivized Nash equilibrium requires that the best response functions have to be computable (see, e.g., Rubinstein (1998)).

In repeated games, one can require that the strategies be computable (Nachbar and Zame (1996); Rubinstein, (1998)) and, as a subgame perfect Nash equilibrium, the best response functions be computable (Nachbar and Zame (1996)). As shown in Nachbar and Zame (1996), the second criterion is in general hard to be satisfied.

Instead of normal form games and repeated games, this chapter concerns infinite games with perfect information.

Gale and Stewart (1956) is the first systematic study of infinite games with perfect information. They showed that all closed games are determined (i.e., one of the players has a winning strategy). The main purpose of this chapter is to introduce an effective version of determinacy for infinite PI-games using the notion of computability (by Turing machines).

The main result of this chapter is a characterization of effective determinacy for closed games.

The rest of the chapter is organized as follows. We sketch the main idea of a Turing machine in the next section. In section 3 we define effective determinacy of a PI-game. Section 4 proves the main result of the chapter.

4.2 Turing Machine

Turing machines were invented by Alan Turing, the father of computer science, in 1936. He wanted to define what an algorithm is in precise mathematical terms. His definition turned out to also be the most useful model of a computer to this date.

Imagine a tape, infinite in both directions, divided into cells. Each cell can contain the symbol "1" or it may be blank. For convenience, let us say that it contains "0" if it is blank. For the time being we should think of the "1" symbol as an uninterpreted vertical scratch and the "0" simply as the absence of a scratch. These may be interpreted as the numerals for one and zero but they need not be.

Now imagine a device with a reading head that can move over the tape or draw the tape through itself. The device scans one cell of the tape at a time. It can do three things (formally called actions) to the tape:

1. read whether the cell being scanned contains "1" or "0",
2. change "1" to "0" and vice versa, and
3. advance to the next cell to the right(R) or left(L) along the tape.

At any given moment device is supposed to be in one of a finite number of internal states $Q = \{q_0, q_1, \cdots, q_n\}$. q_0 is reserved for terminal state.

A program P is a finite (unordered) set of instructions, called quadruples, which tells it what to do (2 and 3 above) depending upon what it finds on the tape (1). Each quadruple must take form

$$q_i S A q_j.$$

Then an instruction $I = q_i SAq_j$ reads: “If the device is in state q_i and the current scanned cell is S (either 0 or 1), then perform the action A (write 1 to the current cell if $SA = 01$, write 0 to the current cell if $SA = 10$; move one cell to the left if $A = L$, to the right if $A = R$) and pass into the new internal state q_j.”

A Turing machine is the device plus the tape plus the program. We follow the convention in identifying a Turing machine M with its program.

To make M perform a computation, we print various symbols on the tape and position the device so that a specified cell is being scanned; further, M must be set in some prescribed initial state. This configuration constitutes the input. Then if M is in the state q_i and scans the symbol S, it acts as described above under an instruction $q_i SAq_j$ in P. This kind of action is then repeated for the new state and symbol scanned, and so on. If M ever enters q_0 during its operation it stops and whatever is printed on the tape at that time is the machine’s output.

When we are dealing with numerical computation we need some effective codings, a way to read (or, to interpret the configuration of the tape). For example we want M to compute a function

$$f : \mathrm{Seq} \to \mathrm{Seq}.$$

Here Seq denotes the set of all finite binary sequences. Then for each input $p = (p(0)), p(1), \cdots, p(n)) \in \mathrm{Seq}$ of f we write $1^\frown p = (1, p(0)), p(1), \cdots, p(n))$ into the tape as part of the configuration and reads the string after the left most 1 in the tape as the output of the computation. We will call this a convention for f. Similarly we can have convention for functions from $\mathbb{N}$ to $\mathbb{N}$, from Seq to $2 = \{0, 1\}$, etc.

We can now define effective computability.

Definition 4.2.1. A function f from a domain A to B is said to be computable if there exists a convention and a Turing machine M such that for each $(a, b) \in A \times B$ and $f(a) = b$ the machine terminates with output b for the input a (read under the convention).

We say that a set is computable if its characteristic function is computable, otherwise it is incomputable.

Proposition 4.2.2. There exists incomputable subsets of $\mathbb{N}$.

In particular the halting problem is undecidable, i.e., the set

$$\{n : n \in W_n\}$$

is incomputable. Here W_n is defined as follows. If we have an effective enumeration of the Turing machines that compute functions from $\mathbb{N}$ to $\mathbb{N}$,

$$\phi_1, \phi_2, \cdots,$$

then W_n is set of all the inputs $m \in \mathbb{N}$ that ϕ_n can compute (i.e., terminates when the input is m).

4.3 Effective Determinacy of PI-games

Let A be a set of infinite binary sequences, i.e., each $f \in A$ takes the form $f = (f(0), f(1), \cdots)$, where each $f(n)$ is either 0 or 1. Associated to the set A an infinite game, G_A, involving two players. The players alternate choosing elements of $\{0, 1\}$

with player I moving first: player I chooses $f(0)$, player II responds by $f(1)$, player I then chooses $f(2)$, etc. Hence an infinite sequence $f = (f(0), f(1), \cdots)$ is specified. Player I wins just in case $f \in A$.

Let Seq be the set of all finite binary sequences. A strategy σ for player I is a subset of Seq such that

1. $\emptyset \in \sigma$;

2. let $(y_0, y_1, \cdots, y_m) \in \sigma$, if m is even, then both $(y_0, y_1, \cdots, y_m, 0)$ and $(y_0, y_1, \cdots, y_m, 1)$ are in σ; otherwise there exists only one element of $\{0, 1\}$, denoted by $\sigma(y_0, y_1, \cdots, y_m)$, such that

$$(y_0, y_1, \cdots, y_m, \sigma(y_0, y_1, \cdots, y_m,)) \in \sigma;$$

3. σ contains no other elements.

A strategy τ for player II can be similarly defined. It is also convenient to regard a strategy as a (partial) function from Seq to $\{0, 1\}$.

If in condition 2 we do not require that $\sigma(y_0, y_1, \cdots, y_m)$ is unique then we get the notion of a quasistrategy.

A strategy σ is a winning strategy for player I if following σ player I always wins, no matter how player II plays. The notion of a winning strategy τ for player II is similarly defined. The game G_A is *determined* if there is a winning strategy for one of the players.

Let $2^{\mathbb{N}}$, the set of all infinite binary sequences, be given the product topology, i.e., the basic neighborhoods are of the form

$$N_p = \{f \in 2^{\mathbb{N}} | f(0) = p(0), \cdots, f(m) = p(m)\},$$

for each $p = (p(0), p(1), \cdots, p(m)) \in \mathrm{Seq}$.

For any $A \subset 2^{\mathbb{N}}$, an $f \in A$ is said to be an isolated point if there exists some $(y_0, y_1, \cdots, y_m) \in \mathrm{Seq}$ such that

$$N_{(y_0,y_1,\cdots,y_m)} \cap A = \{f\}.$$

If f is not an isolated point, then it is called a limit point. Denote by A' the subset of A such that every element is a limit point in A. A set A is called perfect if it is nonempty, closed and $A' = A$.

The games G_A and the notion of determinacy were first introduced by Gale and Stewart (1956). They also proved that all closed games, i.e., G_A such that A is closed, are determined.

We shall now define an effective version of determinacy for the game G_A and prove an analogous determinacy result in the next section.

Recall that a function from natural numbers to natural numbers is said to be computable if there exists a Turing machine that implements it. Let Seq be identified with natural numbers in one of the standard recursive ways. We said that a strategy is computable if it is computable as a function.

Definition 4.3.1. A strategy σ of player I is computable if it is computable as a (partial) function from Seq to $\{0, 1\}$.

Intuitively this corresponds either to the requirement that a strategy can be described in finite terms or to the concept that a strategy should be mechanically implementable. This restriction is interesting both in theory and in practice. Theoretically this models the computational aspects of bounded rationality; practically in many situations the agents playing the game, like computers, machines, robotics, are not capable of playing arbitrary strategies.

We say that player I wins G_A in the effective setting if he possesses a computable strategy σ such that for all computable strategy τ of player II, the resulting play, denoted by $\sigma * \tau$, is in A. The situation for player II is defined in the similar way.

Definition 4.3.2. The game G_A is called *effectively determined* if one of the players wins G_A in the effective setting.

For effective determinacy, we can ask the same question: for which A is G_A effectively determined? It is not hard to give an example that is not effectively determined. This is in sharp contrast with the case of determinacy, since all known examples of indetermined games require axiom of choice.

Example 4.3.3. *Let K, L be two incomparable r.e. sets(see Soare (1987)). Each move of player I in stage $2n$ is intended to be an answer to the question: "Is $n \in K$?" And similarly a move for player II in stage $2n+1$ is intended to be an answer to: "Is $n \in L$?" The payoffs are defined in the following way: If both players answer all the questions correctly, then I wins; otherwise the first player who makes a mistake lose. Since II wins if and only if I makes a mistake earlier than him, each such instance is captured by a finite string p of length l, where l is an odd number, such that for each $2n < l$, $p(2n) = 1$ if and only if $n \in K$; for each $2n-1 < l$, $p(2n-1) = 1$ if and only if $n \in K$; $p(l) = 1$ if and only if $l \notin K$. So the complement of A is the union of all open sets N_p, where p has the above property, hence it is open. Therefore A is a closed set. By the result of Gale and Stewart (1956), it is determined. However, it is not effectively determined. For* *any computable strategy, say of player I, since it is bound to make mistake at some stage (note that K and L are incomparable), say $2n$, then the any strategy of player II answers the first $n+1$ questions regarding L correctly makes player II win the game.*

This shows that G_A need not be effectively determined even for A closed. In the following we shall give a characterization for those closed sets that are effectively determined.

4.4 A Characterization of Effective Determinacy of Closed Games

Lemma 4.4.1. *Let f be an isolated point of A, then the effective determinacy of G_A is equivalent to that of $G_{A\setminus\{f\}}$.*

Proof. Suppose that player I wins $G_{A\setminus\{f\}}$ then it wins G_A too since the payoff set is larger now. Similarly if player II wins G_A, it is going to win $G_{A\setminus\{f\}}$. So we need only prove the other cases.

Suppose that player I wins G_A we will show that he also wins $G_{A\setminus\{f\}}$. Let σ be a winning strategy of player I in G_A. Then for any strategy τ of player II, $\sigma * \tau \in A$. We claim that $\sigma * \tau \in A \setminus \{f\}$, hence I still wins $G_{A\setminus\{f\}}$. Suppose, towards a contradiction, that $\sigma * \tau = f$. Since f is isolated in A, there exists some n such that $N_{f\restriction n} \cap A = f$, where $f \restriction n = (f(0), f(1), \cdots, f(n-1))$. Since σ is a winning strategy for G_A and $\sigma * \tau = f$, the remaining part of σ starting from $f \restriction n$ is a winning strategy for the subgame of G_A by restricting G_A to the part starting from $f \upharpoonright n$. But the payoff of I in this subgame is a singleton f, which he certainly cannot win. This shows a contradiction.

Now assume that II wins $G_{A \setminus \{f\}}$. Since $\bar{A}$ is open, and if f is isolated, there exists some n such that $N_{f \upharpoonright n} \subset \bar{A} \cup \{f\}$. Let τ be a winning strategy for II in $G_{A \setminus \{f\}}$, then it is easy to modify it for a winning strategy in G_A. □

Lemma 4.4.2. *Let S be the set of isolated points of A, then the effective determinacy of G_A is equivalent to that of $G_{A \setminus S}$.*

Proof. Since $2^{\mathbb{N}}$ is compact, the set of isolated points of A is finite. Applying lemma 1 finitely many times the result follows. □

A tree T is a subset of Seq such that if $(p(0), p(1), \cdots, p(m))$ is in T, then $(p(0), p(1), \cdots, p(n))$ is also in T for any $n < m$. If for some $f \in 2^{\mathbb{N}}$ and all m, $f \upharpoonright m \in T$, we say that f is an infinite branch of T. A tree is called finite if the cardinality of T if finite, otherwise it is called infinite. Given any closed set $A \subset 2^{\mathbb{N}}$, we can define a tree T_A representing A by

$$T_A = \{f \upharpoonright n | f \in A \ \& \ n \geq 0\}.$$

Lemma 4.4.3 (König)**.** *If T is infinite, then it contains an infinite branch.*

Proof. See Srivastava (1998). □

Lemma 4.4.4. *If T_A dose not contains any strategy of player I, then II has a computable winning strategy.*

Proof. If T_A dose not contains any strategy, by the determinacy result of Gale and Stewart (1956), II possesses a winning strategy (not necessarily computable), say

τ. Then $\sigma * \tau \notin A$ for each σ. Let $\sigma * \tau = f$, since the complement of A is open, there exists some n such that $N_{f \restriction n}$ is contained in the complement of A. Choose n to be smallest possible. Let F be the collection of all such $f \restriction n$, varying τ. Let T be the smallest tree with all the terminal nodes in F. By König's lemma, T is finite, hence there exists a computable strategy of player II. $\square$

Define a sequence A^α inductively. Let $A^0 = A$. Let A^α be defined. Let $A^{\alpha+1}$ be $(A^\alpha)'$. If λ is a limit ordinal, let $A^\lambda = \cap_{\alpha<\lambda} A^\alpha$.

Lemma 4.4.5. *The effective determinacy of G_A is equivalent to that of G_{A^λ} for any countable limit ordinal λ if this holds for all $\alpha < \lambda$.*

Remark 4.4.6. *The successor step is clear from lemma 2.*

Proof. Suppose that I wins G_A with σ. If I does not win G_{A^λ}, then there exists some τ of II such that $\sigma * \tau = f \notin A^\lambda$. Since for each such f there exists a least α such that $f \notin A^\alpha$, let $\alpha_0 + 1$ be the least of all such α and the f corresponding to $\alpha_0 + 1$ is f_0. σ is still a winning strategy for $G_{A^{\alpha_0}}$, but it fail to be a winning strategy for $G_{A^{\alpha_0+1}}$, as is shown in the proof of Lemma 1, this is impossible.

Now suppose that II wins G_{A^λ} with τ. We modify τ such that it wins over all of G_{A^α}, $\alpha < \lambda$. Let $\tau_0 = \tau$. Let τ_α be defined and τ_α is a (possibly incomputable) strategy such that for all computable strategy σ of player I, $\sigma * \tau_\alpha \neq f_\beta$ for all $\beta < \alpha$.

Search for the least $\gamma < \lambda$ such that there exists a computable strategy σ of I such that $\sigma * \tau = f_\gamma$. If such γ does not exists we are done, let $\tau_\alpha = \tau_\beta$ for all $\alpha < \lambda$. Otherwise define τ_γ as follows.

Since f_γ is an isolated point in A_γ, there exists a least n such that $N_{f_\gamma \restriction n} \cap A_\gamma = \{f_\gamma\}$. Let τ_γ be a strategy such that τ_γ agrees with τ_α everywhere except at the subgame starting from $f_\gamma \restriction n$, where τ_γ avoids f_γ and other f_β such that $f_\beta \restriction n = f_\gamma \restriction n$. This is possible since the collection of f_γ and all f_β is only a countable set (λ is countable) but $N_{f_\gamma \restriction n}$ is uncountable and other f_β are already avoided by τ_α. Define τ_β for all $\alpha < \beta < \gamma$ to be $\tau = \gamma$.

Finally let $\tau_\lambda = \lim\limits_{\alpha<\lambda} \tau_\alpha$. τ_λ is well-defined since each $N_p \cap \tau$ is modified at most finitely many times.

So τ_λ is a strategy of player II such that for any computable strategy σ of I, $\sigma * \tau \notin A$. If τ is computable we are done. Otherwise we can apply lemma 4 to get a computable winning strategy. $\square$

The following theorem guarantees that the iteration terminates.

Lemma 4.4.7 (Cantor-Bendixson)**.** *A^λ is either empty or perfect (i.e., nonempty, closed and dense in itself) for some countable ordinal λ.*

Proof. See Srivastava (1998). $\square$

Let λ_0 be the least such ordinal and write A^∞ for A^{λ_0}.

Lemma 4.4.8. *Let $A \subset 2^{\mathbb{N}}$ be a closed set. Let σ be a computable strategy for player I. Then I wins G_A in the effective setting by σ if and only if $\sigma \subset T_A$.*

Proof. Let σ be a computable strategy such that I wins G_A, i.e. for all computable τ of player II, $\sigma * \tau \in A$. Suppose that $|\sigma| \subsetneq A$, then there is a strategy τ' of player II (need not be computable) such that $\sigma * \tau' \notin A$. Let $\sigma * \tau' = f$, note that the complement of A is open, so there exists n such that $N_{f \restriction n} \cap A = \emptyset$. It is then clear that τ' can be modified to be a computable strategy τ such that $\sigma * \tau \notin A$, a contradiction.

The other direction is clear. $\square$

Lemma 4.4.9. *Let S be a maximal quasistrategy contained in T_A, where A is a perfect set. II wins $G_{|S|}$ if and only if he wins G_A effectively.*

Proof. Suppose τ is a (not necessarily computable) strategy of player II that wins $G_{|S|}$, i.e., for all computable strategy σ of player I, $\sigma * \tau \notin |S|$. Let σ be computable such that $f = \sigma * \tau \in A \setminus |S|$. By maximality of S, there exists a least n such that $T_{f\restriction n}$ contains no strategy of I in the subgame $T_{f\restriction n}$, hence it is possible to modify τ to avoid f and also wins the subgame $T_{f\restriction n}$.

In this manner it is possible to modify τ so that it avoids all of $A \setminus |S|$. By lemma 4 such τ can be chosen to be computable. $\square$

By lemma 7 we know that player I wins if and only if there is a computable strategy $\sigma \subset T_A$. If such a purification does not exist for him, then we have the following.

Lemma 4.4.10. *Player II wins if and only if there exists a computable strategy τ such that for all $\sigma \subset T_A$, $\sigma * \tau$ is incomputable (as a partial function of σ).*

Proof. Let τ be a winning, computable strategy of player II. Suppose that, for some $\sigma \subset T_A$, $\sigma * \tau$ is computable, since $\sigma * \tau \in A$, it is easy to reformulate σ to be a computable strategy that wins against τ; This proves the necessity part.

Now let τ be such that for all $\sigma \subset T_A$, $\sigma * \tau$ is incomputable. Suppose that there exists some σ computable such that $\sigma * \tau = f \in A$. By lemma 8 we can assume

T_A is a quasistrategy. So it is easy to modify σ to a (not necessarily computable) strategy σ' such that σ' so that $\sigma' * \tau = f$. This is possible since $f \in A$. Now f is computable since both σ and τ are. This proves the sufficiency part. $\square$

If neither of them wins, then T_A is a set that has the following property

$$\forall \sigma, \tau (\tau \text{ is computable }, \sigma \subset T_A \implies \sigma * \tau \text{ is computable}). \tag{4.1}$$

So we have proved

Theorem 4.4.11. *G_A is effectively determined if and only if T_{A^∞} does not satisfy (4.1).*

4.5 Conclusion

In this chapter we consider an effective version of determinacy. Not all games are effectively determined, even for closed games. We prove a characterization result for determinacy of closed games.

A paper closely related to this one is Deng and Mahajan (1997). They consider a semi-effective version of determinacy in which only the first player is restricted to using computable strategy. It is easy to see that the analysis in this chapter leads to the same characterization as above with the computability of τ is removed.

Bibliography

[1] Binmore, K. G., M.J. Osborne, and A. Rubinstein (1992), "Non-cooperative Models of Bargaining". In R. J. Aumann S. Hart (eds.)*Handbook of game Theory with Economic Applications*, Amsterdam: North-Holland.

[2] Brânzei, R.(2000), "On the Determinateness of n-person games with information energy", *Revue Roumaine de Mathématiques Pures et Appliquées*, 45: 67-76.

[3] Davis, Morton. (1964), "Infinite Games of Perfect Information", In: Dresher, Melvin, Lloyd S. Shapley, and Alan W. Tuchker (eds.) *Advances in Game Theory, Annals of Mathematical Studies* , 52: 85-101. Princeton, Princeton University Press.

[4] Deng, X. and Mahajan, S. (1997) "The Cost of Derandomization: Computability or Competiveness", *SIAM Journal of Computing*, Vol.26, No.3, 786-802.

[5] Gale, David. (1953), "A theory of n-person games with perfect information", *Proceedings of the National Academy of Sciences U.S.A.*, 39: 496-501.

[6] Gale, D. and F.M. Stewart (1953), "Infinite Games with Perfect Information, Contributions to the Theory of Games", Vol. II, *Annals of Mathematical Studies*, 28: 245-266.

[7] Kanamori, Akihiro. (2000), *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*, Berlin: Springer.

[8] Kechris, A. S. (1995), *Classical Descriptive Set Theory*, New York : Springer-Verlag.

[9] Kuhn, H. W. (1953), "Extensive Games and the Problem of Information, Contributions to the Theory of Games", Vol. II, *Annals of Mathematical Studies*, 28: 193-216.

[10] Löwe, B. "Determinacy for infinite games with more than two players with preferences", *Journal of Applied logic*, Forthcoming.

[11] Martin, D. A. (1975), "Borel Determinacy", *Annals of Mathematics*,102, 263-371.

[12] Maskin, Eric and Tirole, Jean (2001), "Markov Perfect Equilibrium: I. Observable Actions", *Journal of Economic Theory*, 100, 191-219.

[13] Mycielski, J (1992), "Games with Perfect Information". In R. J. Aumann S. Hart (eds.)*Handbook of game Theory with Economic Applications*, Amsterdam: North-Holland.

[14] Nachbar, J. H. and Zame, W. R. (1996) "Non-computable Strategies and Discounted Repeated Games", *Economic Theory*, 8, 103-122.

[15] Nash, J. F. (1950b), “Equilibrium Points in N-Person games”, *Proceedings of the National Academy of Sciences (U. S. A.)*, 36, 48-49.

[16] Osborne, M.J. and A. Rubinstein (1990), *Bargaining and Markets*, Academic Press: San Diego.

[17] Romer, D. (2006), *Advanced Macroeconomics*, 3rd ed., Boston, Mass.: McGraw-Hill.

[18] Rubinstein, A. (1982), “Perfect Equilibrium in a Bargaining Model”, *Econometrica*, 50, 97-109.

[19] Rubinstein, A. (1998), *Modeling Bounded Rationality*, MIT Press, Cambridge, Massachusetts.

[20] Samuelson, P. A. (1958),“An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money,” *Journal of Political Economy*, December 1958, 467-482

[21] Shaked, A. and J. Sutton (1984), “Involuntary Unemployment as a Perfect Equilibrium in a Bargaining Model”, *Econometrica*, 52, 1351-1364.

[22] Selten, R. (1965), “Spieltheoretische Behandlung eines Oligopolmodels mit Nachfragetragheit”, *Zeitschrift fur die Gesamte Staatswissenschaft*, 121, 301-324.

[23] Selten, R. (1975), “Re-examination of the Perfectness Concept for Equilibrium Points in Extensive Games”, *International Journal of game Theory*, 4, 25-55.

[24] Shell, K. (1971), "Notes on the Economics of Infinity," *Journal of Political Economy* , Vol. 79(5), 1002-1011.

[25] Soare, R. I. (1986), *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Princeton, New Jersey.

[26] Srivastava, S.M. (1998), *A Course on Borel Sets*, New York: Springer-Verlag.

[27] Sutton, J. (1986), "Non-cooperative bargaining theory: An introduction", *Review of Economic Studies*,53, 709-724.

[28] von Neumann, J. (1928), "Zur Theorie der Gesellschaftsspiele", *Mathematische Annalen* 100, 295-320. English Translation, Tucker, A.W. and Luce,R.D. ed., "Contributions to the Theory of Games", Vol. IV, *Annals of Mathematical Studies* 40, 1959.

[29] von Neumann, J. and Morgenstein, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey.

[30] Wolfe, Philip. (1955), "The Strict Determinacy of Certain Infinite Games", *Pacific Journal of Mathematics*, 5: 841-847.

[31] Zermelo, E. (1913), 'Uber eine Anwendungen der Mengenlehre auf die Theorie der Schachsiels', *Proceedings of the International Fifth Congress of Mathematicians, Cambridge, 1912, vol. II*, Cambridge: Cambridge University Press, 501-504.